Solid Geometry and Spherical Trigonometry

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PREFACE

This book * was planned with attention to the present national need for increased numbers of men and women who have a substantial appreciation of spatial relations and acquaintance with the fundamentals of spherical trigonometry. However, the brief but logically complete section on solid geometry would be suitable as text material whenever a class is to receive a rounded treatment of this subject in a short period of time. The typical student in view is a freshman in college who failed to study solid geometry in high school and has already studied plane trigonometry. The solid geometry was prepared by Walter W. Hart and the spherical trigonometry by William L. Hart.

THE SOLID GEOMETRY

This section is designed as the basis for a brief course but the content could also serve effectively in a *review* of solid geometry as a preliminary to the study of spherical trigonometry.

The aim is to provide experience in thinking about and constructing three-dimensional figures and to give the student a satisfactory background for the study of spherical trigonometry or other subjects for which a knowledge of solid geometry is an essential prerequisite.

The topics have been selected so as to present an essentially complete cross section of the usual classical treatment of solid geometry, with certain shifts of emphasis.

Brevity has been achieved by treating the subject from a somewhat more mature point of view than has been customary and by providing experience in the independent demonstration of theorems, not by many relatively inconsequential exercises, but rather by suggestions for finishing incomplete proofs of fundamental theorems.

To facilitate the selection of an extremely brief course, the chapter on mensuration is placed after that on the sphere, so that most of the omissions could occur in the final chapter.

THE SPHERICAL TRIGONOMETRY

The theory and formulas for the solution of right and oblique spherical triangles are presented in a very complete fashion.†

* This text is composed of Parts II and III of the authors' *Plane Trigonometry, Solid Geometry, and Spherical Trigonometry.*

† Except as to the use of the *haversine,* although its definition and main utility are indicated.

Plane and middle latitude sailing, the treatment of which is based on the solution of right *plane* triangles, are presented as an appropriate introduction to related applications involving spherical trigonometry.

The applications of spherical triangles on the earth emphasize *distance and direction problems involved in geography and navigation.* These problems, other applications relating to the *celestial sphere,* and supplementary discussion are designed to provide a background for the student who wishes to know the nature of celestial navigation or who will study this subject later. However, no attempt is made in this book to present a treatment of celestial navigation.

For a brief course in spherical trigonometry, the teacher may desire to present the chapter on the right spherical triangle, merely the laws of sines and cosines relating to the oblique triangle, and then the chapter on applications, with the omission of plane and middle latitude sailing.

Answers are included for all problems involving extensive computation; usually, results are given for **both four- and five-place computation.** *Logarithmic and Trigonometric Tables* by William L. Hart are extremely complete so as to permit three-, four-, or five-place computation as desired.

WALTER W. HART
WILLIAM L. HART

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PART III. SPHERICAL TRIGONOMETRY

PART II. SOLID GEOMETRY

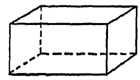
Chapter XII

PLANES AND POLYHEDRAL ANGLES

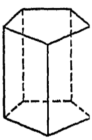
119. The difference between plane and solid geometry. A fundamental figure of plane geometry is the rectangle. It lies on a plane surface or plane. It has length and width, and is called a *two-dimensional* figure.

A straight line is a *one-dimensional* figure.

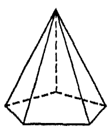
The whole figure in plane geometry always lies on a plane. For this reason, plane geometry is called two-dimensional geometry.



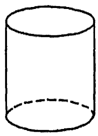
The rectangular solid is a figure of solid geometry. It has length, width, and height; it is *three-dimensional*. Other three-dimensional shapes are represented below.



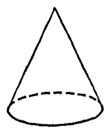
PRISM



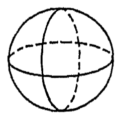
PYRAMID



CYLINDER

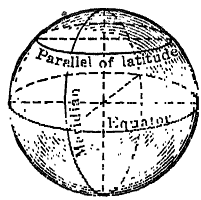


CONE

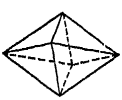


SPHERE

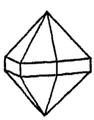
The earth, for practical purposes, and its surrounding space may be considered as essentially spherical, so that navigation in the air as well as on water and the science of astronomy are based on concepts and facts studied in solid geometry.



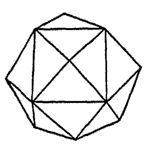
The crystals of certain minerals have the forms that follow; such forms are called polyhedrons.



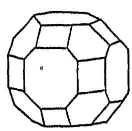
COMMON SALT



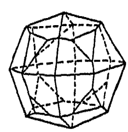
QUARTZ



COPPER

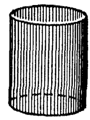


SILVER



GOLD

120. Your answers for the following questions are to be suggested by *your intuitive knowledge* of *plane*, *cylindrical*, *conical*, and *spherical* surfaces, as these are represented in the first set of figures in § 119, and as they occur on such familiar objects as table tops, cylindrical tanks, conical funnels, and baseballs, respectively.



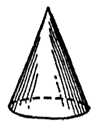
1. (*a*) Are there two points on a cylindrical surface that can be joined by a straight line that lies entirely on the surface?

(*b*) Are there many such pairs of points?

2. (*a*) Are there two points on a cylindrical surface that determine a line that does not lie on the surface?

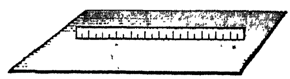
(*b*) Are there many such pairs of points?

3. Consider the line that joins any two points on a spherical surface such as a baseball. Does it lie on the surface, or does it cut through the surface?



4. Repeat both parts of Example 1 if the surface is a conical surface, such as that at the right.

5. Repeat both parts of Example 2 if the surface is a conical surface.



6. If two points of a straightedge lie on a *plane surface*, at how many other points will the straightedge touch the surface?

7. If two points of a straight line lie on a plane, where do all the other points of the line lie?

8. Let a thin flat card represent a plane surface.

(*a*) Can the card be held in more than one position in space and still touch a stationary pencil point?

(*b*) What fact about the number of planes in space that contain one fixed point is suggested by part (*a*)?

9. (*a*) Can a card be held in more than one position in space and still touch *two* fixed points?

(*b*) What fact about the number of planes in space that contain two fixed points is suggested by part (*a*)?

10. (*a*) Can a card be held in more than one position in space and still touch three fixed points, that do not all lie on the same straight line?

(*b*) What fact about the number of planes in space that contain three fixed points, not all on the same straight line, is suggested by part (*a*)?

11. Can a card that passes through one of two parallel lines be held in more than one position in space, and still pass through the other line?

121. (*a*) Points are **collinear** when they lie in the same straight line; they are **co-planar** when they lie in the same plane.

(*b*) A geometric **figure is determined** if one and only one like it contains specified points and (or) lines.

122. A plane is defined by the following *postulates:*

(*a*) *If a straight line joins two points of a plane, it lies entirely in the plane.*

(*b*) *Three non-collinear points determine a plane.*

(*c*) *If two planes have one common point, they have a second and are said to intersect.*

NOTE. "Common point" means a point that lies on each of two lines or surfaces, or on a line and a surface.

123. The following facts *can be proved:*

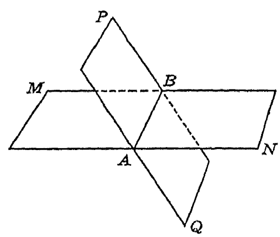
(*a*) *Two intersecting straight lines determine a plane.*

(*b*) *A straight line and a point not on it determine a plane.*

(*c*) *Two parallel lines determine a plane.*

124. The intersection of two surfaces, or of a surface and a line, consists of all points that lie in both.

125. *If two planes intersect, the intersection is a straight line.*



Hyp. Planes MN and PQ have points A and B in common.

Con. Their intersection is a straight line.

Proof. Draw straight line AB.

$\therefore$ AB lies in MN and PQ. Why?

If MN and PQ had any common points outside of AB they would coincide (§ 122, *b*). But, by hypothesis, they are different planes. So AB must be their complete intersection.

1. Why is a tripod used for mounting cameras, telescopes, and surveying instruments?

2. In how many planes can one straight line lie?

3. Are two straight lines in space necessarily parallel if they do not meet?

4. Are four random points in space likely to be co-planar?

5. If a straight line joins two points that lie on opposite sides of a plane, in how many points will it cut the plane?

126. *The following Exercises illustrate lines perpendicular to planes.*

1. Let a sheet of paper on your desk represent a plane MN. On it place a point O. Hold a pencil, with its point at O, in the position which you would consider perpendicular to plane MN.

2. Through O, draw several lines on plane MN. Hold your pencil perpendicular to plane MN at O. What angle does your pencil appear to make with each of the lines through O?

3. Place point X on your plane MN. Through X, draw the line YZ on MN.

(*a*) Can you hold your pencil with its point at X, so that it is perpendicular to YZ but not perpendicular to plane MN?

(*b*) Do you think that a straight line can be perpendicular to a line in a plane without being perpendicular to the plane?

4. Draw a straight line AB; on it place a point C. Hold a pencil so that it represents a line perpendicular to AB at C. In how many positions can you hold your pencil and still satisfy the conditions of the first two sentences?

5. (*a*) Hold one pencil perpendicular to a plane MN at a point B. Try to hold a second pencil perpendicular to MN at B.

(*b*) How many perpendiculars to a plane at a point of the plane *do you think* there can be?

6. (*a*) Hold one pencil perpendicular to plane MN at A, and a second pencil perpendicular to plane MN at B. What kind of lines do these pencils suggest to you?

(*b*) What tentative conclusion about lines perpendicular to the same plane is suggested by part (*a*)?

7. (*a*) Hold the tip of one finger above plane MN. Can you place a pencil touching your finger tip and perpendicular to plane MN?

(*b*) Can more than one line passing through a point outside a plane be perpendicular to the plane?

(*c*) What tentative conclusion is suggested by parts (*a*) and (*b*)?

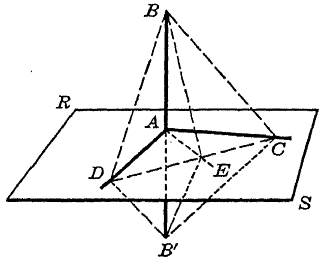
8. (*a*) Draw a line AB and place point C on it. Can you hold a card so that one of its edges will pass through C and so that the card will be what you consider perpendicular to AB?

(*b*) Can the card be in more than one position and still satisfy the conditions of part (*a*)?

(*c*) What tentative conclusion is suggested by parts (*a*) and (*b*)?

127. A straight line is perpendicular to a plane if it is perpendicular to every straight line in the plane, passing through its foot; also the **plane** is **perpendicular to the line.**

128. *If a line is perpendicular to each of two lines at their intersection, it is perpendicular to their plane.*



Hyp. $AB \perp AD$; $AB \perp AC$.
AD and AC determine plane RS.

Con. $AB \perp$ plane RS.

Proof. 1. Draw AE, any other line in RS through A; also DEC, cutting AD, AE, and AC at D, E, and C. Extend BA to B', making $AB' = BA$.

Draw BD, BE, BC, $B'D$, $B'E$, and $B'C$.

2. Then	$DB = DB'$; $CB = CB'$.	App. § 21.
3.	$\triangle DBC \cong \triangle DB'C$.	App. § 11(c).
4.	$\therefore \angle BDE = \angle B'DE$.	Why?
5.	$\triangle BDE \cong \triangle B'DE$.	App. § 11(a).
6.	$\therefore EB = EB'$.	Why?
7.	$\therefore AE$ is a $\perp$-bisector of BB' or $BB' \perp AE$.	App. § 12.

8. Since AE is any line in RS through A (other than AD and AC)

$BAB' \perp$ plane RS. § 127.

NOTE 1. "Constructions" in solid geometry are theoretical. Instead of "draw AE" as in this proof, it would be more appropriate to say *there exists a line* AE, etc. However, it is customary and *is* brief to say *draw*.

NOTE 2. This is a *model proof*. Notice that only one statement appears on a line, that it is numbered, and that the authority for it is suggested by a reference. If the student writes out a proof in the model form, these references should be written *in full* either at the right of or below the statements.

Most of *the proofs* in the text *are only outlined*. If a student wishes to insure mastery of the subject, and has the time, it will pay to write out many or all these proofs in the model form.

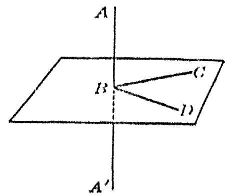
NOTE 3. The reference App. § 21 in Step 2 refers to § 21 in the list of *plane geometry* references found in the Appendix for Part II on pages 212 to 214.

1. If equal oblique lines are drawn to a plane from a perpendicular to the plane, they cut off equal segments from the foot of the perpendicular and make equal angles with the perpendicular.

2. If obliques to a plane from a point on a perpendicular to the plane cut off unequal segments from the foot of the perpendicular, they are unequal.

129. *A plane can be constructed perpendicular to a line:*
(*a*) *At a point of the line.* (*b*) *From a point not on the line.*

Proof of (*a*). If B is on AA', draw $BC \perp AA'$ in any plane through AA'. Also draw $BD \perp AA'$ in any *other* plane through AA'.



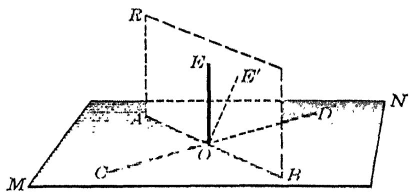
Then plane $CBD \perp AA'$ at B. Why?

Proof of (*b*). If C is not on AA', draw $CB \perp AA'$ in plane CAA'. Then draw $BD \perp AA'$ in any other plane through AA'.

Then plane $CBD \perp AA'$, through C.

Remark. In both cases, it can be proved that *only one plane can be drawn that satisfies the given requirements.*

130. *At a point of a plane a perpendicular can be constructed to the plane.*



Proof. Through O of plane MN, draw any straight line CD in MN. Then draw plane $RB \perp CD$ at O; also draw EO in $RB \perp AB$.

$CD \perp EO$, since $CD \perp RB$. § 127.

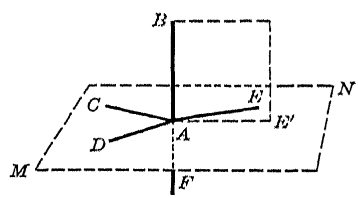
$EO \perp AB$ and CD.

$\therefore$ $EO \perp MN$. § 128.

Remark. EO is the *only perpendicular* to MN at O. A second, like $E'O$, would coincide with EO since both would be $\perp AB$ in RB. App. § 13(*a*).

131. *All the perpendiculars to a straight line at a point of the line must lie in a plane perpendicular to the line at the given point.*

Proof. If CA, DA, and EA are all $\perp BF$ at A, then $AB \perp$ plane CDA. (Why?)



Plane BEA cuts CDA in a line AE'.

$\therefore$ $BA \perp AE'$. Why?

$\therefore$ AE and AE' must coincide since both are in BAE and $\perp BA$ at A.

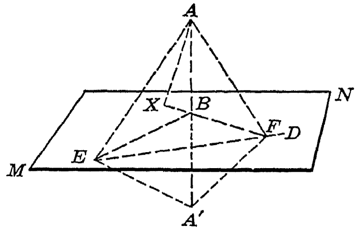
$\therefore$ AE must fall in CDA.

$\therefore$ AC, AD, and AE all lie in CDA.

1. If two obliques, drawn to a plane from a point on a perpendicular to the plane, cut off equal segments from the foot of the perpendicular, they are equal and make equal angles with the perpendicular.

2. Non-collinear points A, B, and C are each equidistant from the ends of segment XY. Prove that A, B, and C determine a plane that is perpendicular to XY at its mid-point.

132. *One and only one perpendicular can be constructed to a plane from a point not on the plane.*



Const. From A, not on plane MN, draw $AF \perp$ at F to any line ED in MN. In MN draw $XF \perp DE$ at F.

In AFX, draw $AB \perp XF$.

Statement. $AB \perp MN$.

Proof. Extend AB to A', making $BA' = AB$. Draw AE, EB, $A'E$, and $A'F$.

$DE \perp$ plane AFA'. § 128.

$\therefore\ ED \perp A'F$ (Why?); also $AF = A'F$. Why?

$\therefore\ \triangle EFA \cong \triangle EFA'$ (Why?); and $EA = EA'$.

$\therefore\ EB \perp AA'$, or $AA' \perp EB$. App. § 12.

$\therefore\ AA' \perp MN$. Why?

A second perpendicular to MN from A like AX is impossible, because both would be in plane AXB and $\perp XB$.

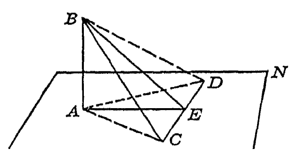
133. *The perpendicular to a plane from a point not on the plane is the shortest segment to the plane from the point.*

Proof. In the figure for § 132, AB and any other line AE to MN from A determine a plane that intersects MN in BE.

AB must be less than AE. Why?

134. The distance from a point to a plane is the perpendicular to the plane from the point.

1. Through the foot of a perpendicular to a plane, a line is drawn perpendicular to any line in the plane, and a line is drawn from the intersection to any point on the given perpendicular. Prove that this line from the intersection is perpendicular to the given line in the plane.



Hyp. $AB \perp MN$; CD is in MN.
$AE \perp CD$ at E.

Con. $BE \perp CD$.

Suggestion. Take $ED = EC$. Prove $AD = AC$; also $BD = BC$.

Then prove $BE \perp CD$.

2. Each of three lines is perpendicular to each of the other two. Prove that each is perpendicular to the plane determined by the other two.

3. A plane is perpendicular to a segment at its mid-point. How do the distances from any point of this plane to the ends of the segment compare?

The following Exercises suggest theorems about parallels.

1. (*a*) Hold two pencils to represent two parallel lines.

(*b*) Do these lines lie in the same plane?

2. (*a*) Hold two pencils to represent two lines in space that do not intersect but are not parallel.

(*b*) Do these lines lie in the same plane?

3. (*a*) Let a card represent plane MN and a second card represent plane RS. Hold planes MN and RS in the position which you would call parallel.

(*b*) What definition of parallel planes do you suggest?

4. (*a*) Let a pencil represent line AB. Hold your pencil and your card to represent a line AB parallel to a plane MN.

(*b*) What definition of "line parallel to a plane" do you suggest?

5. (*a*) Place your plane MN on your desk and hold your line AB parallel to it. Now place your plane RS so that it rests against line AB and intersects plane MN. Mark on plane MN the intersection of it and RS. Call the intersection XY.

(*b*) What kind of lines do AB and XY appear to be?

6. Hold two cards to represent planes MN and RS perpendicular to a line CD drawn on a sheet of paper. What kind of planes do MN and RS appear to be?

7. Draw line CD on a sheet of paper on your desk. Hold AB parallel to CD. What relation does line AB appear to bear to the plane represented by the sheet of paper?

8. Hold two pencils to represent lines parallel to a given plane. Are the pencils necessarily parallel?

9. (*a*) On your plane MN, draw any straight line AB. Hold plane MN parallel to plane RS.

(*b*) What relative position do AB and RS appear to occupy?

10. (*a*) On a sheet of paper, draw line AB. Hold two pencils to represent lines RS and XY, each parallel to AB.

(*b*) What kind of lines do RS and XY appear to be?

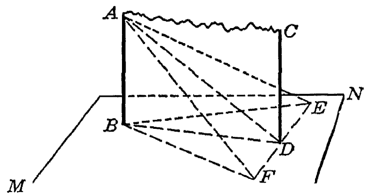
11. Hold a pencil so that its point, P, is above a plane MN. How many lines through P can be parallel to MN?

135. (*a*) **A straight line and a plane are parallel** if they do not meet however far extended.

(*b*) **Two planes are parallel** if they do not meet however far extended.

(*c*) **Two straight lines are skew** if they do not lie in the same plane. They cannot meet however far extended.

136. *If two straight lines are perpendicular to the same plane, they are parallel.*



Hyp. $AB \perp MN$; $CD \perp MN$.

Con. $AB \parallel CD$.

Proof. Draw BD and AD. In MN, draw $FDE \perp BD$, making $FD = DE$. Draw BF, BE, AF, and AE.

Then $BF = BE$. App. § 21.

$\triangle ABF \cong \triangle ABE$. Prove.

$\therefore \quad AF = AE$. Why?

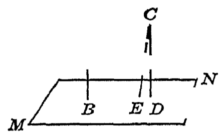
$\therefore \quad AD \perp FE$. App. § 12.

$\therefore \quad CD$, BD, and AD are co-planar. § 131.

$AB \perp BD$; $CD \perp BD$. Why?

$\therefore \quad AB \parallel CD$. App. § 16.

137. *If one of two parallels is perpendicular to a plane, the other is also.*

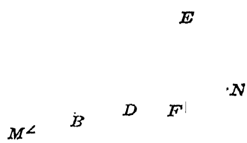


Hyp. $AB \parallel CD$ and $AB \perp MN$.

Con. $CD \perp MN$.

Proof. CD would coincide with the perpendicular CE drawn to plane MN from C, because CE and CD would both be parallel to AB (App. § 15).

138. *If each of two straight lines is parallel to a third line, they are parallel to each other.*



Hyp. $AB \parallel EF$ and $CD \parallel EF$.

Con. $AB \parallel CD$.

Proof. If plane MN is perpendicular to EF, then AB and CD must both be perpendicular to MN. Why?

$\therefore \quad AB \quad CD$. Why?

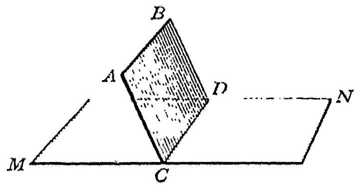
1. *Prove.* If three planes intersect in pairs, their lines of intersection are either concurrent or parallel.

Suggestions. Case 1. Assume that two of the lines of intersection meet at P. Prove that P must lie on the third.

Case 2. Assume that two of the lines of intersection are parallel. Prove that the third must be parallel to each of them, by using an indirect proof, based on Case 1.

2. Prove that any point on the plane perpendicular to a segment at its mid-point is equidistant from the ends of the segment.

139. *If a line outside a plane is parallel to a line of the plane, it is parallel to the plane.*



Hyp. $AB \parallel CD$; CD is in MN.

Con. $AB \parallel MN$.

Proof. AB and CD lie in plane AD. Plane AD intersects MN in CD. § 125.

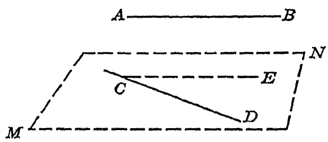
If AB were to intersect MN at a point X, then X would be in AD and in MN. $\therefore$ X would be in CD. § 124.

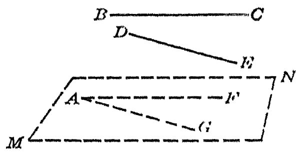
Then AB would meet CD at X, which is impossible.

140. *A plane can be constructed parallel to:*

(*a*) *One of two skew lines through the other.*

(*b*) *Each of two skew lines through a point that is not on either.*

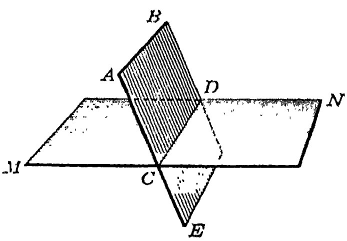




Suggestions. (*a*) Through C of CD, draw CE parallel to AB. Then AB is parallel to plane CDE. § 139.

(*b*) Through the point A, draw $AF \parallel BC$ and $AG \parallel DE$. Then plane GFA must be parallel to BC and DE. Why?

141. *If a straight line is parallel to a plane, it is parallel to the intersection of the plane and any plane drawn through the straight line.*



Hyp. $AB \parallel MN$; AD cuts MN in CD.

Con. $AB \parallel CD$.

Proof. AB and CD are co-planar. AB cannot meet CD, because it would then intersect MN, to which it is parallel.

$\therefore$ $AB \parallel CD$. App. § 14.

1. A straight line, not in either of two given planes, is parallel to the intersection of the planes. Prove that it is parallel to each of the planes.

2. Prove that two lines that are each perpendicular to the same plane are co-planar.

3. Prove that every line in one of two parallel planes is parallel to the other plane.

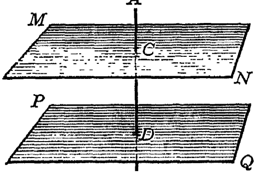
142. *If two planes are perpendicular to the same straight line, they are parallel.* See Fig. (*a*).

Hyp. $MN \perp AB$ at C; $PQ \perp AB$ at D.

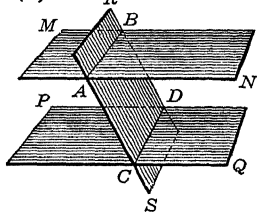
Con. $MN \parallel PQ$.

Suggestion. If MN were to meet PQ and X were any point on their intersection, XC and XD would both be $\perp AB$. This is impossible. (Why?) Therefore MN must be parallel to PQ.

(a)



(b)



143. *Through a point not on a given plane, a plane can be drawn parallel to the given plane.* Use Fig. (*a*).

Suggestion. If C is not on PQ, ACD can be drawn $\perp PQ$, and plane $MN \perp ACD$ at C. Then $MN \parallel PQ$.

Remark. It can be proved that only one plane can be drawn parallel to a given plane through a point not on the given plane.

144. *If two parallel planes are cut by a third plane, their intersections are parallel.* Use Fig. (*b*).

Hyp. $MN \parallel PQ$; RS cuts MN in AB and PQ in CD.

Con. $AB \parallel CD$.

Suggestion. AB and CD are co-planar. They cannot meet because then MN would meet PQ, and this is impossible.

145. *If a straight line is perpendicular to one of two parallel planes, it is perpendicular to the other also.*

Hyp. $MN \parallel PQ$; $AD \perp MN$.

Con. $AD \perp PQ$.

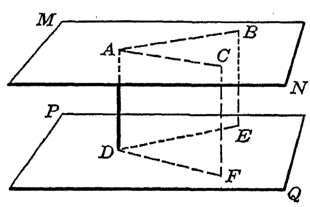
Proof. Draw AB and AC in MN. Let BAD and CAD cut PQ in DE and DF, respectively.

$\therefore$ $DE \parallel AB$; and $DF \parallel AC$. § 144.

$AD \perp AB$ and AC.

$\therefore$ $AD \perp DE$ and DF.

$\therefore$ $AD \perp PQ$. Why?



146. The distance between two parallel planes is the perpendicular distance between them.

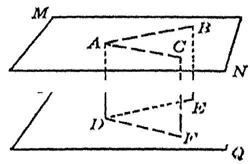
147. *Two parallel planes are everywhere equidistant.*

Hyp. $PQ \parallel MN$; $AD \perp MN$; $CF \perp MN$. (See Fig. § 148.)

Con. $AD = CF$.

Suggestion. Prove AD and $CF \perp PQ$; $AD \parallel CF$.
Then $AC \parallel DF$ (§ 144), and $AD = CF$. App. § 23.

148. *If each of two intersecting lines is parallel to a plane, their plane is parallel to the given plane.*



Hyp. $AB \parallel PQ$; $AC \parallel PQ$.

Con. Plane BAC, or $MN \parallel PQ$.

Proof. Draw $AD \perp PQ$. Then plane CAD cuts PQ in $DF \parallel AC$. Similarly $DE \parallel AB$.

$AD \perp DF$ and $AD \perp DE$. Why?

$\therefore$ $AD \perp AC$; $AD \perp AB$. App. § 17.

$\therefore$ $AD \perp MN$; and $MN \parallel PQ$. Why?

149. *If two angles not in the same plane have their sides respectively parallel and in the same directions from their vertices, they are equal and their planes are parallel.* (See Fig. § 148.)

Hyp. $\angle CAB$ is in MN; $\angle FDE$ is in PQ.
$AC \parallel DF$; $AB \parallel DE$.

Con. (*a*) $\angle CAB = \angle FDE$. (*b*) $MN \parallel PQ$.

Proof. (*a*) Make $AC = DF$, and $AB = DE$. Draw BC, EF, AD, CF, and BE.

Then $ACFD$ is a $\square$. [App. § 24(*a*).] $\therefore$ $CF = AD$; $CF \parallel AD$.

Similarly $BE = AD$; and $BE \parallel AD$.

$\therefore$ $CF = BE$; and $CF \parallel BE$; and then $CB = EF$. Prove.

$\therefore$ $\triangle ABC \cong \triangle DEF$; and $\angle CAB = \angle FDE$. Prove.

(*b*) $AC \parallel DF$. $\therefore$ $AC \parallel PQ$. (§ 139.) Similarly $AB \parallel PQ$.

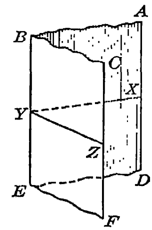
$\therefore$ $MN \parallel PQ$. Why?

1. Through a line parallel to a plane, planes are passed intersecting the given plane. What is true about the lines of intersection?

2. From a parallel to a plane, parallel segments are drawn to the plane. Prove that these segments are equal.

3. A line is parallel to each of several parallel planes. Through it a plane is passed intersecting each of the parallel planes. What kind of lines are the intersections?

150. (*a*) **When two planes intersect,** the parts of the planes on one side of their line of intersection form a **dihedral angle;** as $\angle C\text{-}BE\text{-}A$. Its **edge** is BE; its **faces** are $BEDA$ and $CBEF$.



The faces extend infinitely far.

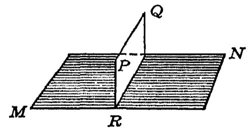
(*b*) **A plane angle of a dihedral angle** is formed by two rays, one in each face, perpendicular to the edge at the same point; as $\angle ZYX$.

(*c*) **Two dihedral angles are equal** when their plane angles are equal.

(*d*) **The measure of a dihedral angle** is the measure of its plane angle.

(*e*) A dihedral angle is **right, acute,** or **obtuse** according as its plane angle is right, acute, or obtuse. Two dihedral angles are **complementary** or **supplementary** according as their plane angles are complementary or supplementary.

(*f*) **Two planes are perpendicular** if they form right dihedral angles; or if the adjacent dihedral angles formed are equal.



151. *If two planes intersect:*

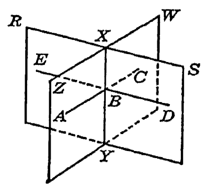
(*a*) *The opposite or vertical dihedral angles are equal.*

(*b*) *Adjacent dihedral angles are supplementary.*

Hyp. Planes RYS and ZYW intersect in line XY.

Con. (*a*) $\angle RXYZ = \angle SXYW$.

(*b*) $\angle RXYZ + \angle ZXYS = 180°$.



Proof. (*a*) Draw ABC in $ZYW \perp XY$ at B; and EBD in $RYS \perp XY$ at B.

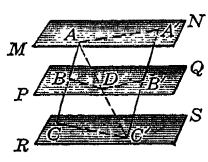
Then $\angle EBA$ is the plane angle of $\angle RXYZ$; $\angle ABD$ of $\angle ZXYS$, and $\angle DBC$ of $\angle SXYW$. § 150(*b*).

$\angle EBA = \angle DBC$. (App. § 7.) $\therefore$ $\angle RXYZ = \angle SXYW$. § 150(*c*).

(*b*) $\angle EBA + \angle ABD = 180°$. App. § 8.

$\therefore$ $\angle RXYZ + \angle ZXYS = 180°$. § 150(*e*).

1. Prove that three parallel planes intercept proportional segments on two transversals.

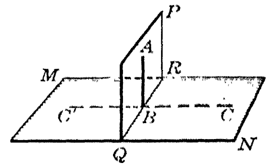


Hyp. $MN \parallel PQ \parallel RS$, cutting AC in A, B, and C respectively, and $A'C'$ in A', B', and C'.

Con. $AB : BC = A'B' : B'C'$.

Suggestion. Use the adjoining figure.

152. *If two planes are perpendicular, a straight line in one of them perpendicular to their intersection is perpendicular to the other.*



Hyp. $PQ \perp MN$ and cuts it in QR.
AB in PQ is $\perp QR$.

Con. $AB \perp MN$.

Proof. Draw CBC' in $MN \perp QR$.
$AB \perp QR$. Why?
$\therefore \angle ABC$ is the plane $\angle$ of $\angle PQRN$. § 150(b).
$PQ \perp MN$. $\therefore \angle ABC = 90°$. § 150($f$).
$\therefore AB \perp BC$. $\therefore AB \perp MN$.

153. *If a straight line is perpendicular to a plane, any plane through it also is perpendicular to the plane.*

Hyp. $AB \perp MN$ at B. PQ through AB cuts MN in QBR.

Con. $PQ \perp MN$.

Suggestion. Draw CBC' in $MN \perp QR$.
$\therefore \angle ABC$ is the plane angle of $\angle PQRN$. Why?
$\angle ABC = 90°$. Why?
$\therefore PQ \perp MN$. Why?

154. *If two planes are perpendicular, a perpendicular to one of them at a point of their intersection lies in the other.*

Suggestion. If $PQ \perp MN$ and $AB \perp MN$ at a point B of the intersection QR of PQ and MN, then AB must lie in PQ, because a line $A'B$, in PQ drawn $\perp QR$ at B, would be $\perp MN$, and must coincide with AB (§ 130, Remark).

155. *If two planes are perpendicular, a perpendicular to one of them from a point of the other must lie in the other.*

Suggestion. If $PQ \perp MN$ and $AB \perp MN$ from point A in PQ, then AB must lie in PQ because a line AX, in PQ, drawn $\perp QR$, is $\perp MN$, and must coincide with AB (§ 132).

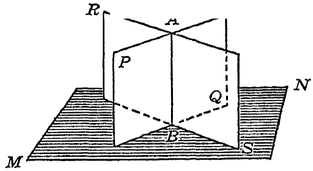
$\therefore$ AB also must lie in PQ.

1. *Prove.* A plane perpendicular to the edge of a dihedral angle is perpendicular to the faces of the angle.

2. *Prove.* Perpendiculars to the faces of a dihedral angle from a point within the angle must lie in a plane that is perpendicular to the edge of the angle.

3. *Prove.* A plane can be constructed that bisects a given dihedral angle; also, any point on this plane is equidistant from the faces of the given dihedral angle.

156. *A plane perpendicular to each of two intersecting planes is perpendicular to their intersection.*



Hyp. $MN \perp RS$; and $MN \perp PQ$.
RS cuts PQ in AB.

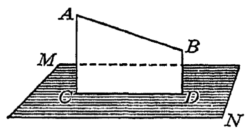
Con. $MN \perp AB$.

Suggestion. A line AX, from A, $\perp MN$, must lie in RS and PQ. § 155.
$\therefore AB$ coincides with AX and is $\perp MN$.

157. *Through a straight line not perpendicular to a plane, one and only one plane can be drawn perpendicular to the given plane.*

Hyp. AB is not $\perp MN$.

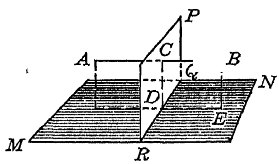
Con. One and only one plane can be drawn through $AB \perp MN$.



Suggestion. Draw $AC \perp MN$. Then plane $BAC \perp MN$. If two *different* planes through AB were $\perp MN$, their intersection also would be. But AB is their intersection and is not $\perp MN$.

1. Perpendiculars are drawn to the faces of a dihedral angle from a point within the angle. Prove that they form an angle that is the supplement of the dihedral angle.

2. *Prove.* If a straight line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.

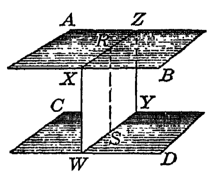


If $AB \parallel MN$ and PR, $\perp AB$, cuts MN in RQ, then $PR \perp MN$.

Suggestion. Draw CD in $PR \perp RQ$. Let plane BCD cut MN in DE.
$\therefore DE \parallel CB$ (§ 141). $\therefore DE \perp PR$. § 137.
Now prove $\angle CDE$ is plane $\angle$ of $\angle PQRN$, and 90°.

3. Given an $\angle AOB$. Let plane RS be perpendicular to AO and plane MN be perpendicular to BO. Prove that the intersection of planes RS and MN is perpendicular to the plane of $\angle AOB$.

4. *Prove.* A plane perpendicular to one of two parallel planes is perpendicular to the other also.

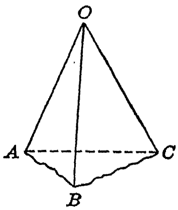


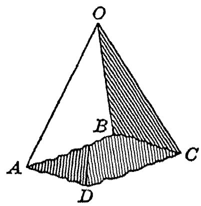
Hyp. $AB \parallel CD$; $XY \perp AB$.

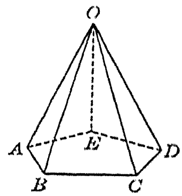
Con. $XY \perp CD$.

Suggestion. Draw RS, in XY, $\perp XZ$. Then $RS \perp AB$; $RS \perp CD$; $XY \perp CD$.

158. (*a*) **A polyhedral angle** is formed when three or more intersecting planes have only one common point; as $O\text{–}ABC$, $O\text{–}ABCD$, and $O\text{–}ABCDE$.







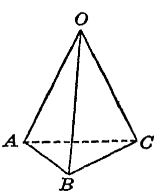
The **faces** of the angle are the parts of the planes that form the angle; as OAB, OAC, and OBC of $\angle O\text{–}ABC$. The **vertex** is the common point; as O. The **edges** are the intersections of the faces; as OA, OB, and OC of $\angle O\text{–}ABC$. The **face angles** are the angles formed by the edges; as $\angle AOB$, $\angle BOC$, and $\angle COA$ of $\angle O\text{–}ABC$. The **dihedral angles** of the polyhedral angle are the angles formed by the faces; as $\angle C\text{–}OA\text{–}B$, $\angle A\text{–}OB\text{–}C$, and $\angle B\text{–}OC\text{–}A$ of $\angle O\text{–}ABC$.

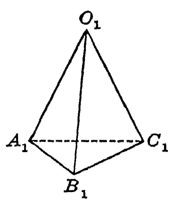
(*b*) A polyhedral angle is **convex** if the polygon formed by a plane that intersects all the faces on one side of the vertex is convex; as $O\text{–}ABCD$, above. Only such will be studied in this text.

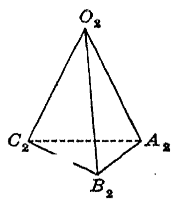
(*c*) **A trihedral angle** is a polyhedral angle having three faces; as $O\text{–}ABC$.

(*d*) Polyhedral angles are **congruent** if their parts are equal, each to each, and arranged in the same order; they are **symmetric** if the parts are arranged in opposite orders.

Thus: $O\text{–}ABC$ and $O_1\text{–}A_1B_1C_1$ are congruent;
$O\text{–}ABC$ and $O_2\text{–}A_2B_2C_2$ are symmetric.

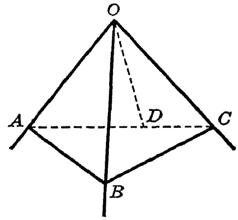






(*e*) Two polyhedral angles are **vertical** when the faces of one are extensions of the faces of the other, through a common vertex. It can be proved that they are symmetric.

159. *The sum of two face angles of a trihedral angle is greater than the third.*



Hyp. $\angle AOC > \angle AOB$; $\angle AOC > \angle BOC$.

Con. $\angle AOB + \angle BOC > \angle AOC$.

Proof. On face AOC, draw OD making

$\angle AOD = \angle AOB$, and $OD = OB$.

Draw plane ABC through B and D, cutting the faces in $\triangle ABC$.

Then $\triangle AOB \cong \triangle AOD$, and $AB = AD$.

$AB + BC > AD + DC$. (Why?) $\therefore$ $BC > DC$. App. § 28. Ax. 10.

In $\triangle BOC$ and $\triangle COD$:

$OC = OC$; $OB = OD$; $BC > DC$.

$\therefore$ $\angle BOC > \angle DOC$. App. § 31.

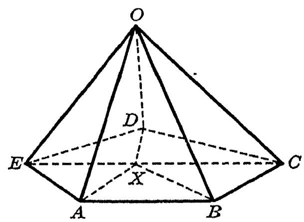
$\therefore$ $\angle AOB + \angle BOC > \angle AOD + \angle DOC$, or $\angle AOC$. Ax. 9.

160. *The sum of the face angles of any convex polyhedral angle is less than* 360°.

Hyp. $O\text{–}ABCDE$ is a convex polyhedral angle.

Con. $\angle AOB + \angle BOC + \angle COD + \angle DOE + \angle EOA < 360°$.

Proof. Let a plane cut the faces in polygon $ABCDE$. Join X, inside $ABCDE$, to each vertex.



In $\angle A\text{–}BEO$, $\angle EAO + \angle OAB > \angle EAB$. (§ 159.)

In $\angle B\text{–}ACO$, $\angle ABO + \angle OBC > \angle ABC$.

In $\angle C\text{–}BDO$, $\angle BCO + \angle OCD > \angle BCD$.

In $\angle D\text{–}CEO$, $\angle CDO + \angle ODE > \angle CDE$.

In $\angle E\text{–}DAO$, $\angle DEO + \angle OEA > \angle DEA$.

Adding, the sum of the base ∠s of the △s with vertex O is greater than the sum of the base ∠s of the △s with vertex X. Ax. 12.

But the sum of all the angles of each set of triangles is the same; namely, $5 \times 180°$, or 900°. Why?

$\therefore$ the sum of the angles at $O <$ the sum of the angles at X. Ax. 13.

The sum of the angles at $X = 360°$. App. § 9.

$\therefore$ the sum of the angles at $O < 360°$. Why?

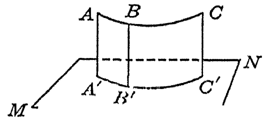
1. Can the face angles of a trihedral angle measure:

(*a*) 90°, 100°, and 130°? (*b*) 80°, 150°, and 70°?

2. Prove that any face angle of *any* polyhedral angle is less than the sum of the remaining face angles.

161. (*a*) **The projection of a point on a plane** is the foot of the perpendicular to the plane from the point.

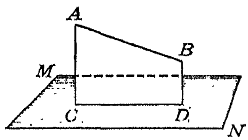
Thus: A' is the projection of A on MN if AA' is perpendicular to MN.



(*b*) **The projection of a line on a plane** consists of the projections on the plane of all the points of the line.

Thus: $A'B'C'$ is the projection of ABC on MN if it consists of the projections on MN of all the points of the curve ABC.

162. *The projection on a plane of a straight line that is not perpendicular to the plane is a straight line.*

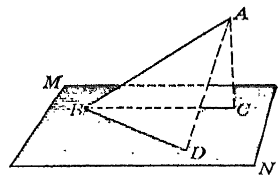


Hyp. AB is not $\perp$ plane MN.

Con. The projection of AB on MN is a straight line.

Suggestion. Through AB, pass plane AD perpendicular to MN. Prove that the feet of all the perpendiculars to MN from points on AB lie on the line CD (§ 155).

163. *The acute angle that a straight line makes with its projection on a plane is the least angle that it makes with any line in the plane, through its foot.*



Hyp. BC is the projection of AB on MN. BD is any other line in MN passing through B.

Con. $\angle CBA < \angle DBA$.

Suggestion. Let $BD = BC$.
Compare: (*a*) AD and AC;
(*b*) $\angle CBA$ and $\angle DBA$, using App. § 31.

164. (*a*) **The inclination of a straight line to a plane** is the acute angle made by it and its projection on the plane.

(*b*) *The projection of a segment on a plane equals the length of the segment multiplied by the cosine of the inclination of the segment to the plane.*

Suggestion. $BC \div BA = \cos \angle CBA$. $\therefore$ $BC = ?$

1. If two parallels are oblique to a plane, their projections on the plane are parallel or coincide.

2. If two segments are equal and parallel, their projections also are equal and parallel.

1. How long is the projection of AB on MN:

(*a*) If $AB = 10$ inches and $\angle CBA = 30°$?
(*b*) If $AB = 10$ inches and $\angle CBA = 45°$?
(*c*) If $AB = 10$ inches and $\angle CBA = 60°$?

2. If AB remains unchanged, how does its projection on MN change when its inclination to MN increases?

3. Let $\angle CBA = 50°$. What is the projection of AB on MN if:

(*a*) $AB = 10$ in.? (*b*) $AB = 15$ in.? (*c*) $AB = 20$ in.?

4. How does the projection of AB on MN change if its inclination to MN is constant and AB increases?

5. A segment is 25 inches long. How long is its projection on a plane if its inclination to the plane is 40°?

6. How does the projection of AB on MN compare with AB:

(*a*) When AB is parallel to MN?
(*b*) When AB makes with MN an angle of 60°?

7. If a segment is perpendicular to a plane, how long is its projection on the plane?

8. Can the projection on a plane of a curved line or of a broken line be a straight line?

9. If two parallels meet a plane, they make equal angles with the plane.

10. A segment is perpendicular to one of two perpendicular planes. Prove that its projection on the other plane is perpendicular to the intersection of the planes.

11. One side of a right angle is parallel to a plane. Prove that the projection of the sides of the angle on the plane is also a right angle.

12. $\triangle ABC$ is located above a plane MN. AB is parallel to MN, and altitude CD to AB makes an angle of 30° with MN. If $AB = 18$ in., and $CD = 12$ in., find the area of $\triangle ABC$ and also the area of its projection on plane MN.

13. If two parallel segments are oblique to a plane, they have the same ratio as their projections on the plane.

14. If a straight line intersects two parallel planes, it makes equal angles with the planes.

15. Is the projection on a plane of a parallelogram also a parallelogram?

16. If two equal segments are drawn to a plane from a point not on the plane, they make equal angles with their projections on the plane and their projections are equal.

17. A segment is parallel to one of two perpendicular planes. Prove that its projection on the other plane is parallel to the intersection of the two planes.

What conclusion follows from the facts given in each of the following Exercises? Why? Sketch a figure if necessary.

1. Plane RS cuts plane MN in AB and plane XY in CD. $MN \parallel XY$.

2. Points R and S lie on both of planes MN and PQ.

3. Plane MN contains AB and cuts plane XY in CD. AB is parallel to plane XY.

4. Line CD contains points X and Y of plane MN.

5. Line $AB \perp$ plane MN and plane $RS \perp$ line AB.

6. Line AB and line AD are in plane MN. AB and AD are parallel to plane XY.

7. Line AB is in plane MN. Point X is on AB.

8. Line $AB \parallel$ line XY. AB, but not XY, is in plane MN.

9. Plane $MN \perp$ plane RS. MN cuts RS in CD.

(*a*) Line XY in MN is $\perp CD$ at X.

(*b*) Line $XY \perp RS$ at O on CD.

(*c*) XY in $MN \perp CD$ at Y; ZY in $RS \perp CD$ at Y.

10. Line $AB \perp$ plane XY. AB is in plane MN.

11. In trihedral $\angle O\text{–}XYZ$, $\angle XOY = 40°$ and $\angle YOZ = 60°$.

12. Plane $XY \perp$ plane MN. A is in XY. Line $AB \perp MN$.

13. X is 12 in. from A and 12 in. from B. Plane $RS \perp AB$ at C. $AC = BC$.

14. Lines XY, RY, and AY are all $\perp$ line CD at Y.

15. Line $AB \perp$ plane MN. Plane $RS \parallel$ plane MN.

16. Line $RS \parallel$ line TW and plane $MN \perp RS$.

17. Line AC is in planes MN and PQ. Plane $XY \perp MN$ and $XY \perp PQ$.

18. X is 5 in. from plane MN and 5 in. from plane RS. Plane MN cuts plane RS.

19. Line $AB \perp$ plane MN and plane $MN \perp$ line CD.

20. Line $XY \parallel$ line AB and line $AB \parallel$ line RS.

21. Line $AB \perp$ line XY at Y and $\perp$ line ZY at Y.

22. On line AB, $AC = CB$. Plane $MN \perp AB$. C is on MN. D is on MN and is 10 in. from A.

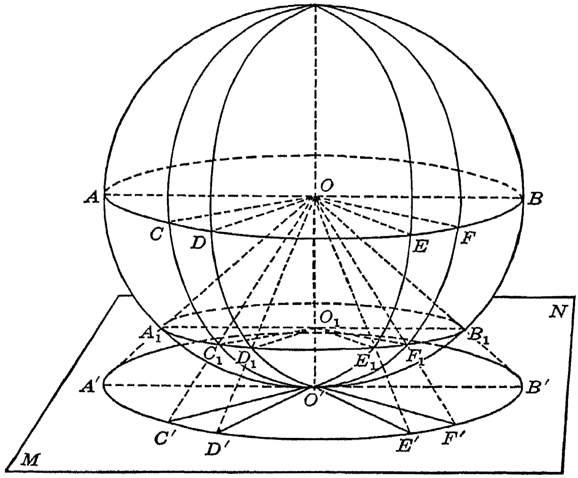
23. Lines AB and CD are oblique to a plane MN. Lines AX and CY are perpendicular to MN.

24. Planes MN and XY are cut by plane CD.

(*a*) Plane MN is parallel to plane XY.

(*b*) Plane MN is perpendicular to plane XY.

25. Plane $MN \parallel$ plane PQ. Line XY is the edge of dihedral $\angle WXYZ$. Planes MN and PQ intersect XY and the faces of $\angle WXYZ$.



Chapter XIII

SPHERICAL GEOMETRY

165. (*a*) **A sphere** is a closed surface whose points are all equidistant from a point, called the **center.**

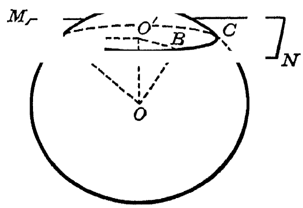
The figure above, from an early page of a text on navigation, represents a sphere, and some of the circles and lines associated with it, projected on a plane that is *tangent* to the sphere at its "south pole."

(*b*) **A radius of a sphere** is a straight line segment from the center of the sphere to a point of the sphere.

(*c*) **A diameter of a sphere** is a straight line segment through the center of the sphere, having its end points on the sphere.

(*d*) *All radii and all diameters of the same sphere are equal.*

166. *If a plane intersects a sphere, the intersection is a circle.*



Hyp. A and B are any points of the intersection of sphere O and plane MN.

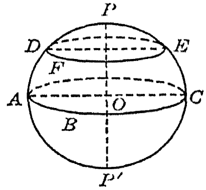
Con. The intersection is a circle.

Proof. Draw $OO' \perp MN$ at O'. Draw OA, OB, $O'A$, and $O'B$.

$\triangle OO'A \cong \triangle OO'B$; $O'A = O'B$. Prove.

$\therefore$ all points of the intersection lie on a circle with radius $O'A$ and center O'. App. § 5.

167. (*a*) **A great circle of a sphere** is the intersection of the sphere and a plane through its center.



The *center* of a great circle is the center of the sphere; the *radius* is the radius of the sphere; as OA. Therefore:

All great circles of a sphere are equal.

A great circle of a sphere separates the sphere into two equal surfaces, **hemispheres.**

The planes of two great circles intersect in a diameter of the sphere, since each passes through the center.

(*b*) **A small circle of a sphere** is the intersection of the sphere and a plane that does not pass through the center of the sphere; as $\odot DFE$ above. Its radius is less than that of the sphere.

(*c*) **The axis of a circle of a sphere** is the diameter of the sphere that is perpendicular to the plane of the circle.

(*d*) **The poles of a circle** of a sphere are the ends of its axis.

Thus: POP' is the axis, and P and P' are the poles of circles ABC and DFE.

168. *The following facts about circles of a sphere can be proved.*

(*a*) *Through the ends of a diameter of a sphere an infinite number of great circles can be drawn.*

(*b*) *Through two points not the ends of a diameter, one and only one great circle can be drawn.*

(*c*) *Through a point of a sphere one and only one small circle can be drawn, parallel to a given great circle.*

(*d*) *Through three points of a sphere, in general, one and only one small circle can be drawn.*

(*e*) *A point of a sphere is a pole of one and only one great circle, but of an infinite number of small circles.*

(*f*) *All circles of a sphere that lie in parallel planes have the same axis and the same poles.*

(*g*) *A great circle that passes through one pole of a given circle passes through the other pole also.*

1. Prove that all circles made by parallel planes have the same axis and poles.

2. Prove that a point can be the pole of only one great circle, but of an infinity of small circles.

169. Great circles of a sphere correspond to straight lines in a plane. The following theorem is proved in higher courses in mathematics.

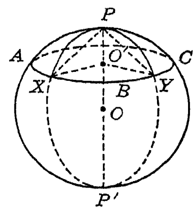
The arc of a great circle, less than a semicircle, between two points of a sphere, is less than any other line on the sphere joining the two points.

170. **The spherical distance** between two points of a sphere is the smaller arc of the great circle between them.

171. *The spherical distances of all points of a circle of a sphere from a pole of the circle are equal.*

Hyp. P and P' are the poles of $\odot ABC$ on sphere O.

Con. All points of $\odot ABC$ are equidistant from P and P'.



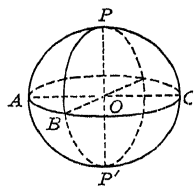
Proof. Let X and Y be any two points of $\odot ABC$. Draw great circle arcs $\widehat{PX}$ and $\widehat{PY}$.

	$POP' \perp$ plane ABC at O'.	§ 167(c) and (d).
Then	$\triangle PXO' \cong \triangle PYO'$.	App. § 11(a).
	$\therefore\ PX = PY;\ \widehat{PX} = \widehat{PY}$.	App. § 32(a).
	$\therefore\ \widehat{P'X} = \widehat{P'Y}$.	Why?

172. The **polar distance** of a circle of a sphere is the spherical distance to any point of it from its nearer pole.

173. *The polar distance of a great circle is a quadrant.*

In the adjoining figure, if P is a pole of great circle ABC, then $\angle POB$ is 90°, and $\widehat{PB}$ must be a quadrant of $\odot PBP'$.



Remark. Hereafter "quadrant" will mean a quadrant of a great circle.

1. (a) What is the length of all great circles of a sphere of which the radius is 10 inches?

(b) What then is the polar distance of great circles of this sphere?

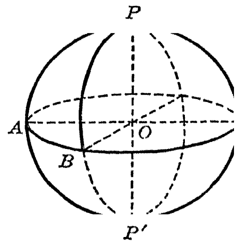
2. The radius of the earth is about 4000 miles. What is the polar distance of any point on the equator?

3. In § 171, if $PO = 10''$, and $\angle XOP = 60°$, what is the polar distance of $\odot ABC$?

4. *Prove.* The planes of all equal small circles of a sphere are equidistant from the center of the sphere.

5. *Prove.* Circles of a sphere that are equidistant from the center of the sphere are equal.

174. *A point of a sphere at a quadrant's distance from each of two other points of the sphere, not the ends of a diameter, is a pole of the great circle through the points.*



Hyp. P is on sphere O. $\overset{\frown}{PA}$ and $\overset{\frown}{PB}$ are quadrants. $\odot ABC$ is the great circle through A and B.

Con. P is a pole of $\odot ABC$.

Proof. Draw PO, OA, and OB. Then $\angle POA = 90°$, and $PO \perp OA$. App. § 35.

Similarly $PO \perp OB$.

$\therefore$ $PO \perp$ plane ABC.

$\therefore$ P is a pole of $\odot ABC$.

1. What is the radius of the small circle that is 3″ from the center of a sphere having radius of 6″?

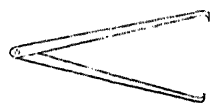
2. In Example 1, what is the length of the chord of a quadrant of a great circle?

3. In Example 1, what is the polar distance of the small circle, correct to tenths of an inch?

4. What is the length of the chord of the polar distance of a great circle on a sphere of radius 12″, correct to tenths?

5. What is the polar distance of the great circle in Example 4?

175. *To draw a great circle on a sphere:.* (*a*) It is necessary to know the radius of the sphere. When the radius is not given, it can be found by means of a pair of outside *calipers*, such as are pictured at the right.



(*b*) Next, find the chord of a quadrant, by drawing on a plane a circle having the radius of the sphere. Find the chord of a quadrant of this circle.

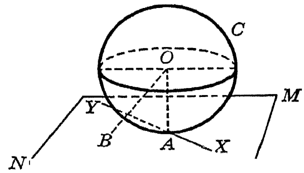
(*c*) With radius equal to the chord of a quadrant and a *pole* of the great circle as center, draw a circle on the sphere. This can be proved to be the required circle.

176. *To locate the pole of a given great circle:* Take as radius the chord of a quadrant. Then, from any two points of the circle as centers, strike intersecting arcs. The intersection is the required pole (§ 174).

If there is a blackened sphere available, carry out the directions for the constructions given in this and the previous section.

177. A line or a plane is tangent to a sphere when it has only one point in common with the sphere.

178. *If a line or a plane is perpendicular to a radius of a sphere at its outer extremity, it is tangent to the sphere.*



Hyp. (*a*) $MN \perp$ radius OA at A.
(*b*) $XY \perp OA$ at A.

Con. (*a*) MN is tangent to sphere O.
(*b*) XY is tangent to sphere O.

Proof. (*a*) Join any point B of MN, except A, with O. Then $OB > OA$.

$\therefore$ B lies outside the sphere.

$\therefore$ MN is tangent to the sphere.

(*b*) The same proof is used for part (*b*).

Remark. The converses of parts (*a*) and (*b*) are obviously true.

179. *All lines tangent to a sphere at a point of the sphere lie in the plane tangent to the sphere at the point.*

Suggestion. Recall § 131.

1. State the converses of parts (*a*) and (*b*) of § 178.

2. A point P lies at distance d from the center of the sphere with radius r (d is more than r). A tangent to the sphere is drawn from P. Express the length of this tangent in terms of d and r.

3. (*a*) How many tangents can be drawn to a sphere from a point outside the sphere?

(*b*) Prove that all these tangents are equal.

4. (*a*) Prove that the points of contact of all the tangents in Example 3 form a circle.

(*b*) What is the radius of the circle in part (*a*)?

NOTE. These tangents form a conical surface. (See § 239.)

5. A great circle and a small circle intersect at right angles, at point P. AB is tangent to the great circle at P, and lies in its plane; CD is tangent to the small circle at P, and lies in its plane. Prove that AB is perpendicular to CD.

Suggestion. What kind of planes are the planes of the two circles? Then use § 152.

6. Let NOS be the axis of great circle XYZ of sphere O; let plane MN be tangent to the sphere at S; let NXS and NYS be two great circles of sphere O, through diameter NOS. Prove that the planes of ⊚NXS and NYS intersect MN in $\angle X'SY'$ that equals $\angle XOY$.

180. (*a*) **A spherical angle** is formed by two great circle arcs; as $\angle BAB'$ in § 183.

(*b*) **The measure of a spherical angle** is defined to be the measure of the angle formed by the tangents to its sides at its vertex; as $\angle DAD'$ in § 183.

(*c*) It is *agreed to say and to write* that *two angles are equal*, or that an angle equals an arc, when their measures are equal. It is implied that the angles are measured in *angular-degrees* and the arcs in *arc-degrees*. This is customary.

181. *A spherical angle has the same measure as the dihedral angle formed by the planes of its sides.*

This is true because the angle formed by the tangents to the sides of the spherical angle at its vertex is the plane angle of the dihedral angle. § 150(*b*) and (*d*); § 180(*b*).

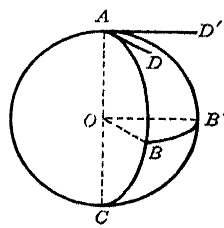
182. *An arc of a great circle drawn to another through a pole of the latter is perpendicular to the latter.*

Thus: In Fig. § 183, $\widehat{AB'}$ must be $\perp$ $\widehat{BB'}$, by § 181.

183. *A spherical angle has the same measure as the arc of the great circle drawn from its vertex as pole and included between its sides, extended if necessary.*

Hyp. Great ⊙s ABC and $AB'C$ lie on sphere O. A is a pole of great circle arc BB'.

Con. $\angle BAB' = \widehat{BB'}$. See § 180(*c*).



Proof. ⊙s ABC and $AB'C$ intersect in diameter AOC. Draw OB and OB'; also AD tangent to $\widehat{AB}$ and AD' tangent to $\widehat{AB'}$.

$\widehat{AB}$ and $\widehat{AB'}$ are quadrants. § 173.

Then $AO \perp OB$ and OB'; or

$OB \perp AO$ and $OB' \perp AO$. App. § 35.

Also $AD \perp AO$, and $AD' \perp AO$. App. § 34.

$\therefore$ $AD \parallel OB$ and $AD' \parallel OB'$. $\therefore$ $\angle BOB' = \angle DAD'$. § 149.

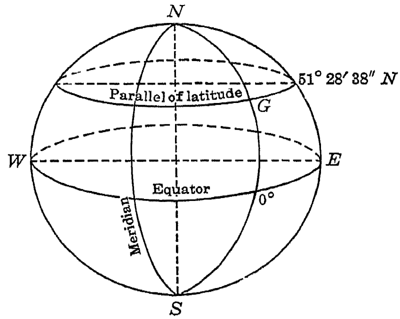
$\angle BOB' = \widehat{BB'}$; $\angle BAB' = \angle DAD'$. Prove.

$\therefore$ $\angle BAB' = \widehat{BB'}$.

1. In a figure like that for § 183, how many degrees are there in $\angle BAB'$ when $\widehat{BB'}$ equals: (*a*) 60°? (*b*) 90°?

2. If $\widehat{BB'}$ increases until it becomes a semicircle, what happens to $\angle BAB'$?

184. The earth is *almost* spherical in shape, its diameter through the equator being about 27 miles more than through its north and south poles. We shall ignore this small variation from spherical shape, although it must be taken into account in certain problems of navigation.



The earth *revolves* once each day around one of its diameters, called its **axis.** The north and south poles of the earth are the ends of its axis.

The equator of the earth is the great circle whose plane is perpendicular to the axis of the earth. Therefore the north and south poles of the earth are also the poles of the equator.

The location of points on the earth is expressed by a set of *meridians* and a set of *parallels of latitude.*

The **principal meridians** are 180 great circles through its poles that divide the equator into 360 one-degree arcs. The *prime meridian* is the one through Greenwich, England; the others, east and west of it, are numbered consecutively from 1° to 180°. Each degree is divided into 60′ and each minute into 60″. The **longitude** of places on earth is expressed by means of these meridians.

Thus: The longitude of New York is 73° 57′ 30″ *W* and that of Leningrad is 30° 17′ 51″ *E*. The difference of longitude of these two places is the sum of the numbers, or 104° 15′ 21″.

The **parallels of latitude** are small circles parallel to the equator, that divide the prime meridian (and every other meridian) into 1° arcs. These parallels are numbered consecutively from 1° to 90° north and south of the equator. The **latitude** of places on the earth is indicated by means of these parallels.

Thus: The latitude of Chicago is 41° 50′ 1″ *N* and that of Rio de Janeiro is 22° 54′ 24″ *S*.

Since the earth revolves through 360° in 24 hours, it revolves (from west to east) 15° in one hour. Therefore two points whose difference in longitude is 15° differ one hour in solar time, the one at the west being one hour behind the one at the east. This variation in solar time led to the introduction of standard time.

1. Prove that the latitude of a place equals its angular distance from the equator.

2. Prove that the difference of longitude of two places is the measure of the dihedral angle formed by the planes of their meridians.

3. Prove that the north and south poles of the earth are the poles of all the parallels of latitude.

4. Prove that every meridian is perpendicular to the equator and to every parallel of latitude.

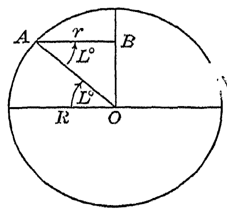
5. What is the difference in solar time of A and B if the longitude of A is $24° 40' W$ and of B is $54° 40' W$?

A nautical mile is the length of one minute of arc of the equator or of any meridian. (See first paragraph of § 184.) It measures about 6080.27 ft. Distances in navigation are expressed in nautical miles.

6. What is the length of the equator (in nautical miles)?

7. What is the distance between two points that have the same longitude if their difference in latitude is 8°?

8. What is the distance between two points on the equator if their difference in longitude is 15°?



9. (*a*) If R represents the radius of the earth, and A is a position having latitude $L° N$, prove that the radius of the parallel of latitude through A is $R \cos L$.

(*b*) What is the length of the parallel in nautical miles?

10. Consider two points that have latitude $L° N$ and difference in longitude of 30°.

(*a*) Using the result of Example 9(*b*) what is the distance between these two points, *measured along the parallel?*

(*b*) Along what line would the shortest distance between the two points be measured?

(*c*) What can you infer then about the shortest distance between the two points?

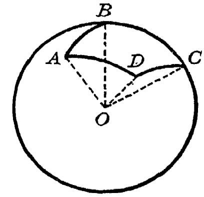
NOTE. This distance can be found by plane trigonometry.

11. (*a*) When it is noon at Greenwich, what time is it at points whose longitude is $180° W$?

(*b*) How many nautical miles west of the prime meridian is such a point if it lies on the equator?

NOTE. Such points lie on the International Date Line; on opposite sides of this line there is a difference of one in the day of the week.

185. (*a*) **A spherical polygon** is formed, as polygon $ABCD$, if three or more points on the same hemisphere, no three on the same great circle, are joined two at a time by great circle arcs.

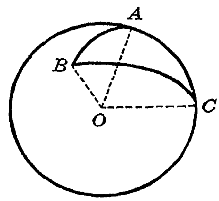


(*b*) **The polyhedral angle** at the center of the sphere, whose edges are the radii drawn to the vertices of the polygon:

(1) Has face angles with measures the same as the corresponding sides of the polygon; as $\angle AOD = \overset{\frown}{AD}$.

(2) Has dihedral angles with measures the same as the corresponding angles of the polygon; as $\angle DAOB = \angle DAB$.

186. *Any side of a spherical triangle is less than the sum of the other two sides.*



Hyp. $\triangle ABC$ is on sphere O.

Con. $\overset{\frown}{AC} < \overset{\frown}{AB} + \overset{\frown}{BC}$.

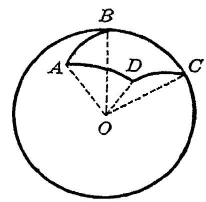
Proof. Form trihedral $\angle O\text{–}ABC$.

Then $\angle COA < \angle BOA + \angle BOC$. § 159.

$\angle COA = \overset{\frown}{CA}$; $\angle BOA = \overset{\frown}{BA}$; $\angle BOC = \overset{\frown}{BC}$.

$\therefore$ $\overset{\frown}{AC} < \overset{\frown}{AB} + \overset{\frown}{BC}$. Why?

187. *The sum of the sides of any convex spherical polygon is less than* 360° *on a great circle of the sphere.*



Hyp. $ABCD$ is a convex spherical polygon on sphere O.

Con. $\overset{\frown}{AB} + \overset{\frown}{BC} + \overset{\frown}{CD} + \overset{\frown}{DA} < 360°$.

Proof. Form polyhedral $\angle O\text{–}ABCD$.

$\angle AOB + \angle BOC + \angle COD + \angle DOA < 360°$. § 160.

$\angle AOB = \overset{\frown}{AB}$; $\angle BOC = \overset{\frown}{BC}$; etc. Why?

(Complete the proof.)

1. What is the maximum length, in inches, of the perimeter of any convex spherical polygon on the sphere of radius 10″?

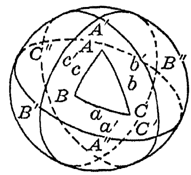
2. (*a*) A spherical triangle has one angle that intercepts 45° on the great circle whose pole is the vertex of the angle. How large is the angle?

(*b*) If the sides of this angle extend to the great circle, how large are they and the other two angles of the triangle?

3. What part of the length of a great circle of a sphere is the perimeter of the spherical triangle whose corresponding face angles measure:

(*a*) 60° each? (*b*) 90° each? (*c*) 90°, 90°, and 120°?

188. Polar triangles. If the vertices of one spherical triangle are the poles of the sides of another, then the second triangle is the **polar triangle** of the first. However, this polar triangle must be selected as follows:



Let $\triangle ABC$ be the given spherical triangle; let A, B, and C be the poles of three great circles. These great circles divide the sphere into eight spherical triangles, of which the polar of $\triangle ABC$ is selected and named thus:

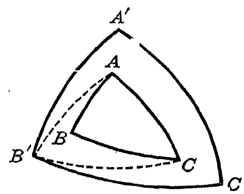
$\overset{\frown}{A'C'}$ and $\overset{\frown}{A'B'}$, drawn from B and C respectively as poles, intersect in two points, of which the one nearer to A is marked A', and the other A''. Similarly B' and B'', C' and C'' are located and marked. Then $\triangle A'B'C'$ is the polar of $\triangle ABC$.

As in plane geometry, $\overset{\frown}{AB}$, $\overset{\frown}{AC}$, and $\overset{\frown}{BC}$ are named $\overset{\frown}{c}$, $\overset{\frown}{b}$, and $\overset{\frown}{a}$; also $\overset{\frown}{A'B'}$, $\overset{\frown}{B'C'}$, and $\overset{\frown}{A'C'}$ are named $\overset{\frown}{c}'$, $\overset{\frown}{a}'$, and $\overset{\frown}{b}'$.

$\overset{\frown}{a}'$ is said to correspond to A; $\overset{\frown}{b}'$ to B; etc.

Notice that A' and A'' are diametrically opposite points of the sphere; also B' and B''; and C' and C''.

189. *If one spherical triangle is the polar of a second, then the second is the polar of the first.*



Hyp. $\triangle A'B'C'$ is the polar of $\triangle ABC$.

Con. $\triangle ABC$ is the polar of $\triangle A'B'C'$.

Plan. We must prove that A' is the pole of $\overset{\frown}{BC}$, B' of $\overset{\frown}{AC}$, and C' of $\overset{\frown}{AB}$. To do so, we prepare to use § 174.

Proof. $\triangle A'B'C'$ is the polar of $\triangle ABC$.

$\therefore$ $\overset{\frown}{AB'}$ or $\overset{\frown}{B'A}$ is a quadrant.

Similarly $\overset{\frown}{CB'}$ or $\overset{\frown}{B'C}$ is a quadrant.

$\therefore$ B' is the pole of AC.

Similarly C' is the pole of $\overset{\frown}{AB}$, and A' of $\overset{\frown}{BC}$.

$\therefore$ $\triangle ABC$ is the polar triangle of $\triangle A'B'C'$.

190. Mutually polar triangles are two spherical triangles on the same sphere, each of which is the polar of the other.

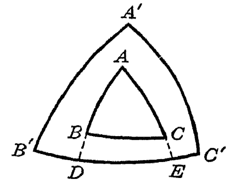
1. Draw the figure for § 189 when:

(*a*) $\overset{\frown}{BC} = 90°$. (*b*) $\overset{\frown}{AB} = \overset{\frown}{BC} = 90°$. (*c*) $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{AC} = 90°$.

191. *In two mutually polar triangles, each angle of one is the supplement of the corresponding side of the other.*

Hyp. $\triangle ABC$ and $\triangle A'B'C'$ are mutually polar.

Con. $\angle A + \widehat{a'} = 180°$; $\angle B + \widehat{b'} = 180°$; $\angle C + \widehat{c'} = 180°$.



Proof. Let $\widehat{AB}$ cut $\widehat{B'C'}$ at D and $\widehat{AC}$ cut $\widehat{B'C'}$ at E.

Then B' is a pole of $\widehat{ACE}$; $\widehat{B'E} = 90°$.

Similarly $\widehat{C'D} = 90°$. § 173.

$\therefore \quad \widehat{B'E} + \widehat{C'D} = 180°.$

$\therefore \quad \widehat{B'E} + \widehat{EC'} + \widehat{DE} = 180°,$

or $\widehat{B'C'} + \widehat{DE} = 180°.$

$\angle A = \widehat{DE}$ (§ 183); $\therefore \quad \widehat{B'C'} + \angle A = 180°.$

$\therefore \quad \angle A + \widehat{a'} = 180°.$ Similarly $\angle B + \widehat{b'} = 180°$; etc.

192. *The sum of the angles of a spherical triangle is more than* 180° *and less than* 540°.

Hyp. $\triangle ABC$ is any spherical triangle.

Con. $\angle A + \angle B + \angle C > 180°$ and $< 540°$.

Proof. Let $\triangle A'B'C'$ be the polar of $\triangle ABC$.

$\therefore \quad \angle A + \widehat{a'} = 180°$; $\angle B + \widehat{b'} = 180°$; and $\angle C + \widehat{c'} = 180°$.

$\therefore \quad (\angle A + \angle B + \angle C) + (\widehat{a'} + \widehat{b'} + \widehat{c'}) = 540°.$ Why?

$\widehat{a'} + \widehat{b'} + \widehat{c'} < 360°.$ § 187.

$\therefore \quad \angle A + \angle B + \angle C > 180°.$ Ax. 13.

Since $\widehat{a'} + \widehat{b'} + \widehat{c'} > 0°$, $\angle A + \angle B + \angle C < 540°$. Why?

193. The spherical excess of a *spherical triangle* is the excess of the sum of its angles over 180°; that of a *spherical polygon* is the excess over $(n - 2)180°$.

1. The sides of a spherical triangle measure 77°, 123°, and 95°. How large are the corresponding angles in the polar triangle?

2. The angles of a spherical triangle measure 86°, 131°, and 68°. How large are the corresponding sides of the polar triangle?

3. If a spherical triangle is equilateral, what is true about the polar triangle?

4. The dihedral angles of a trihedral angle at the center of a sphere measure 75°, 100°, and 65°. How large are the sides of the polar triangle of the intercepted spherical triangle?

1. A spherical polygon lies on a sphere having a radius of 12 in. Prove that the perimeter of this polygon must be less than 24π in.

2. (*a*) In spherical $\triangle ABC$, $\angle B$ and $\angle C$ are each right angles and side BC measures 35°. How large is $\angle A$?

(*b*) How long are $\widehat{AB}$, $\widehat{BC}$, and $\widehat{AC}$ if the radius is 10 in.?

3. What is the spherical excess of the triangle whose angles measure:
(*a*) 90° each? (*b*) 80°, 100°, and 60°? (*c*) 105°, 135°, and 150°?

4. The sides of one spherical triangle measure 75°, 125°, and 95°, respectively.

(*a*) How large are the angles of the polar triangle?

(*b*) What is the spherical excess of the polar triangle?

5. Two sides of a spherical triangle are quadrants and the third side is less than a quadrant.

(*a*) What can you prove about the angles of this triangle?

(*b*) What can you prove about the angles of the polar triangle?

6. A spherical triangle is *bi-rectangular* when *just two* of its angles are right angles.

(*a*) What point is the pole of the side of this triangle that joins the vertices of the two right angles?

(*b*) If this triangle is on the earth, what is its perimeter in nautical miles?

(*c*) What is the relation between the third angle of such a triangle and the side opposite it?

(*d*) If the third angle measures 20°, and the triangle is on the earth, what is the length of the third side in nautical miles?

7. A spherical triangle is *tri-rectangular* when *each* of its angles is a right angle.

(*a*) What is the pole of each of its sides?

(*b*) What is the relation between each side and the opposite angle?

(*c*) If this triangle is on the earth, what is the length of each side in nautical miles?

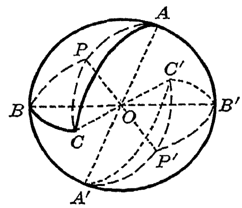
(*d*) If this triangle is on a sphere having 21″ radius, what is its perimeter, correct to tenths of an inch?

8. In spherical $\triangle ABC$ let $\angle B = 15°$, $\angle A = \angle C = 90°$. Draw the figure that represents this triangle and also its polar triangle. Then determine the size of each angle and each side of the polar triangle.

9. If two angles of a spherical triangle are right angles, prove that the corresponding angles of the polar triangle are right angles.

10. In spherical $\triangle XYZ$, $\angle X = 75°$ and X is the pole of $\widehat{YZ}$. If the radius of the sphere is 20 in., how long is each side of $\triangle XYZ$, correct to tenths of an inch?

194. *Diametrically opposite spherical triangles are equal.*



Proof. In the adjoining figure, $\triangle ABC$ and $\triangle A'B'C'$ are located so that AOA', BOB', and COC' are diameters.

Let P be the pole of the small circle through A, B, and C. Draw diameter POP', and great circles $BPB'P'$, $PCP'C'$, and $APA'P'$.

Since $\angle BOP = \angle B'OP'$, $\overset{\frown}{BP} = \overset{\frown}{B'P'}$. Why?

Similarly $\overset{\frown}{PC} = \overset{\frown}{P'C'}$, and $\overset{\frown}{PA} = \overset{\frown}{P'A'}$.

$\overset{\frown}{PB} = \overset{\frown}{PC} = \overset{\frown}{PA}$. § 171.

$\therefore\ \overset{\frown}{PB} = \overset{\frown}{PA} = \overset{\frown}{PC} = \overset{\frown}{P'B'} = \overset{\frown}{P'A'} = \overset{\frown}{P'C'}$. Why?

Dihedral $\angle B\text{–}PO\text{–}C$ = dihedral $\angle B'\text{–}P'O\text{–}C'$. § 151 ($a$).

$\therefore\ \angle BPC = \angle B'P'C'$. § 181.

$\therefore$ $\triangle PBC$ can be made to coincide with $\triangle P'B'C'$, with $\overset{\frown}{PC}$ on $\overset{\frown}{P'B'}$, $\angle P$ on $\angle P'$, and $\overset{\frown}{PB}$ on $\overset{\frown}{P'C'}$; their areas are equal.

Similarly $\triangle PCA = \triangle P'C'A'$, and $\triangle PBA = \triangle P'B'A'$.

Adding, $\triangle ABC = \triangle A'B'C'$. Why?

NOTE. Observe $\triangle ABC$ equals $\triangle A'B'C'$ in area; they are not congruent.

195. If two spherical triangles have the parts of one equal respectively to those of the other:

(a) The triangles are **congruent** if the parts are arranged in the same orders.

(b) They are **symmetric** if the parts are arranged in opposite orders; as $\triangle ABC$ and $\triangle A'B'C'$, above.

196. From the proof of § 194 it is clear that:

(a) *Two diametrically opposite spherical triangles are symmetric.*

(b) *Two symmetric spherical triangles are congruent if one of them is isosceles;* as $\triangle PBC$ and $\triangle P'B'C'$.

They were proved congruent by superposition. Any pair of symmetric spherical triangles, of which one is isosceles, can be made to coincide.

(c) *Any two symmetric spherical triangles are equal.*

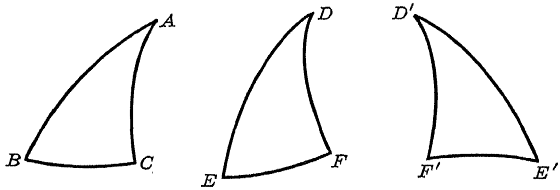
Proof. If a $\triangle A'B'C'$ is symmetric to $\triangle ABC$, form $\triangle A''B''C''$ diametrically opposite $\triangle A'B'C'$. The parts of $\triangle A''B''C''$ must then be equal to those of $\triangle ABC$ as well as those of $\triangle A'B'C'$, and arranged in the same order as those of $\triangle ABC$ since both triangles are symmetric to $\triangle A'B'C'$. $\triangle A''B''C''$ is congruent to $\triangle ABC$, and therefore *equal* to it.

Therefore $\triangle ABC = \triangle A'B'C'$.

197. *Two spherical triangles on the same or equal spheres are congruent, or symmetric and equal, according as the parts are in the same or in opposite orders:*

(*a*) *If two sides and the included angle of one equal two sides and the included angle of the other.*

(*b*) *If two angles and the included side of one equal two angles and the included side of the other.*



Proof of (*a*). If $\overset{\frown}{AB} = \overset{\frown}{DE}$, $\overset{\frown}{AC} = \overset{\frown}{DF}$, and $\angle A = \angle D$, then $\triangle ABC$ can be made to coincide with $\triangle DEF$ just as in plane geometry.

If $\overset{\frown}{AB} = \overset{\frown}{D'E'}$, $\overset{\frown}{AC} = \overset{\frown}{D'F'}$, and $\angle A = \angle D'$, the triangles cannot be made to coincide since the equal parts are in opposite orders. In this case, consider $\triangle A'B'C'$, symmetric to $\triangle ABC$. Its parts, then, must be equal to those of $\triangle D'E'F'$ and arranged in the same order.

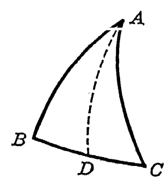
$\therefore \quad \triangle A'B'C' \cong \triangle D'E'F'$, by part (*a*).

$\therefore \quad \triangle ABC$ is symmetric and equal to $\triangle D'E'F'$.

Proof of (*b*). The same procedure can be followed in both cases.

Remark. Observe "equal" in § 197. Equal in such cases means *equal in area* (or in volume, etc.). *Congruent* refers to two figures that can be made to coincide. Congruent figures are equal; equal figures may not be congruent. Also, see § 196(*c*).

198. *If two sides of a spherical triangle are equal, the angles opposite also are equal.*



Thus: If $\overset{\frown}{AB} = \overset{\frown}{AC}$, then $\angle B = \angle C$.

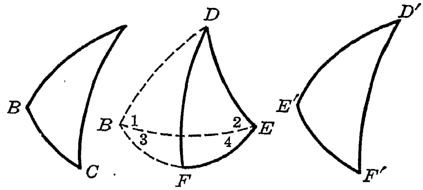
Proof. $\overset{\frown}{AD}$ can be drawn to bisect $\angle A$.

Then $\triangle BDA$ is symmetric to $\triangle CDA$, by part (*a*) of § 197.

$$\therefore \quad \angle B = \angle C.$$

1. *Prove.* If great circle arcs be drawn from any point on a meridian to points on the equator that are equidistant from the intersection of the meridian and the equator, these arcs must be equal and make equal angles with the equator and the meridian.

199. *Two spherical triangles on the same or equal spheres are congruent or symmetric if they are mutually equilateral, depending on the arrangement of the corresponding parts.*



Hyp. $\widehat{AB} = \widehat{DE} = \widehat{D'E'}$;
$\widehat{AC} = \widehat{DF} = \widehat{D'F'}$;
$\widehat{BC} = \widehat{EF} = \widehat{E'F'}$.

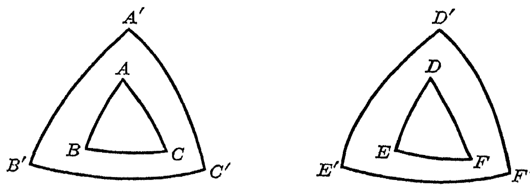
Con. (*a*) $\triangle ABC$ is symmetric to $\triangle DEF$;
(*b*) $\triangle ABC \cong \triangle D'E'F'$.

Proof. (*a*) Place $\triangle ABC$ so that $\widehat{AC}$ coincides with its equal $\widehat{DF}$ and draw great circle arc $\widehat{BE}$.

Then $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$. Hyp.; § 198.
$\therefore \quad \angle 1 + \angle 3 = \angle 2 + \angle 4$. Why?
$\therefore \quad \triangle ABC$ is symmetric to $\triangle DEF$. Prove.

(*b*) In the case of $\triangle ABC$ and $\triangle D'E'F'$, imagine $\triangle D''E''F''$ symmetric to $\triangle D'E'F'$. Therefore it must be symmetric to $\triangle ABC$ by part (*a*).
$\therefore \quad \triangle ABC \cong \triangle D'E'F'$ since both are symmetric to $\triangle D''E''F''$.

200. *Two spherical triangles on the same or equal spheres are congruent or symmetric if they are mutually equiangular.*



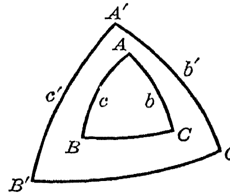
Proof. Let $\triangle ABC$ and $\triangle DEF$ be mutually equiangular. Let $\triangle A'B'C'$ and $\triangle D'E'F'$ be their polar triangles.

$\therefore$ $\triangle$s $A'B'C'$ and $D'E'F'$ are mutually equilateral. § 191.
$\therefore$ $\triangle A'B'C'$ and $\triangle D'E'F'$ are congruent or symmetric. § 199.
$\therefore$ $\triangle A'B'C'$ and $\triangle D'E'F'$ are mutually equiangular. § 195.
But $\triangle ABC$ and $\triangle DEF$ are the polars of $\triangle A'B'C'$ and $\triangle D'E'F'$. § 189.
$\therefore$ $\triangle ABC$ and $\triangle DEF$ are mutually equilateral. § 191.

Being mutually equilateral, by proof, and mutually equiangular, by hypothesis, $\triangle ABC$ and $\triangle DEF$ are either congruent or symmetric, according to the arrangement of their parts.

1. Give the theoretic construction of the arc of a great circle that bisects a given spherical angle.

201. *If two angles of a spherical triangle are equal, the sides opposite are equal.*



Proof. Let $\angle B = \angle C$.

Let $\triangle A'B'C'$ be the polar of $\triangle ABC$.

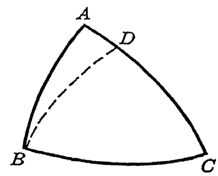
Since $\angle B = \angle C$, then $\overset{\frown}{b'} = \overset{\frown}{c'}$. Why?

$\therefore \quad \angle B' = \angle C'$. § 198.

But $\overset{\frown}{b}$ and $\overset{\frown}{c}$ are the supplements of $\angle B'$ and $\angle C'$ respectively. § 188; § 191.

$$\therefore \quad \overset{\frown}{b} = \overset{\frown}{c}, \quad \text{or} \quad \overset{\frown}{AC} = \overset{\frown}{AB}.$$

202. *If two angles of a spherical triangle are unequal, the sides opposite are unequal, the one opposite the greater angle being the greater; and conversely.*



Hyp. $\angle B > \angle C$. **Con.** $\overset{\frown}{AC} > \overset{\frown}{AB}$.

Proof. Draw great $\odot \overset{\frown}{BD}$, making $\angle CBD$ equal $\angle C$.

Then: $\overset{\frown}{BD} = \overset{\frown}{CD}$; $\overset{\frown}{AD} + \overset{\frown}{BD} > \overset{\frown}{AB}$.

$\therefore \quad \overset{\frown}{AD} + \overset{\frown}{DC} > \overset{\frown}{AB}$, or $\overset{\frown}{AC} > \overset{\frown}{AB}$.

The converse is proved by the indirect method.

203. You observe, then, that there is a theorem about spherical triangles that corresponds to each of several theorems about plane triangles; with *the additional theorem* that two spherical triangles are congruent or symmetric when they are mutually equiangular, whereas two plane triangles are only similar under these conditions.

On the other hand, there is no great circle on a sphere that is parallel to another great circle. Therefore all the theorems of plane geometry that depend on the concept of two parallel lines fail to be true in spherical geometry.

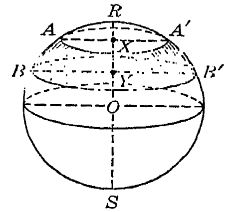
A notable instance is the theorem in § 192. There it was proved that the sum of the angles of a *spherical* triangle is greater than 180°, whereas in plane geometry the sum of the angles of a triangle is 180°.

1. *Prove* that the arc of the great circle bisecting the vertex angle of an isosceles spherical triangle is perpendicular to the base and bisects the base.

2. *Prove* that two points equidistant from the ends of a great circle minor arc determine the great circle perpendicular to and bisecting the arc.

3. *Prove* that an arc can be constructed from a point on a given arc that makes a given angle with the given arc.

204. (*a*) **A zone of a sphere** is the part of the sphere between two parallel planes.

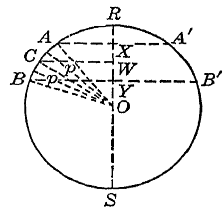


(*b*) The circles, AA' and BB', in which the planes cut the sphere are the **bases** of the zone; and the perpendicular between the planes is its **altitude**; as XY.

(*c*) The zone between R and circle AA' is a *zone of one base.*

205. A *zone may be generated* by revolving an arc of a great circle around the diameter of the great circle.

Thus: Let $RBSB'$ be a great circle, of which $\widehat{AB}$ is an arc. When $RBSB'$ revolves around ROS, $\widehat{RBS}$ generates the sphere and $\widehat{AB}$ generates the zone whose bases have diameters AA' and BB'.



206. (*a*) We find the **area** of a zone by inscribing broken line ACB in $\widehat{AB}$, with $\widehat{AC} = \widehat{CB}$. When $\widehat{AB}$ revolves, AC and BC each generate a frustum of a right circular cone. It can be proved that the area of this frustum is $S = 2\pi p_1 XY$ where p_1 is the distance of chord AC and of chord CB from O.

If we double the number of chords in AB by bisecting $\widehat{AC}$ and $\widehat{CB}$, and revolve the resulting broken line, the area generated by the line is $S = 2\pi p_2 XY$; *p_2 is longer than p_1, as the chords are shorter.*

If we double the number of chords in AB several times, the resulting area obtained becomes approximately that of the zone, and p becomes approximately r. We conclude that:

$$\text{the area, } Z\text{, of the zone} = 2\pi rXY.$$

$$\therefore \quad \boldsymbol{Z = 2\pi rh.}$$

(*b*) The *formula for the area of the sphere* is obtained by observing that the sphere is the zone whose altitude is $2r$.

$$\therefore \quad \text{the area, } S\text{, of the sphere} = 2\pi r(2r).$$

$$\therefore \quad \boldsymbol{S = 4\pi r^2.}$$

Thus: If the radius of the sphere is 14, the area of the sphere is

$$4 \times \frac{22}{\cancel{7}} \times \overset{2}{\cancel{14}} \times 14, \text{ or } 2464.$$

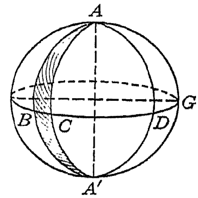
207. (*a*) **A lune** is that part of a sphere that lies between two semicircles of great circles.

(*b*) **The angle of a lune** is the angle between the semicircles.

208. It can be proved that:

(*a*) *Lunes of a sphere having equal angles are congruent, and therefore equal (or equal in area).*

(*b*) *Lunes of the same sphere have the same ratio as their angles.*



Thus: If $\angle BAC = \frac{1}{2}\angle DAG$, then

$$\text{lune } BACA' = \tfrac{1}{2} \text{ lune } DAGA'.$$

(*c*) A sphere may be considered the lune whose angle is 360°. Therefore *the area of a lune whose angle contains A degrees is to the area of the sphere as A is to* 360.

(*d*) If we let L_A = the area of the lune whose angle contains A degrees, then $L_A = A(4\pi r^2)/360$.

$$\therefore \quad \boldsymbol{L_A = \pi r^2 A/90}.$$

209. As a consequence of § 208(*b*):

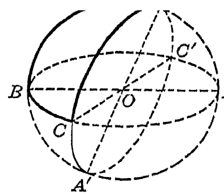
$$L_A + L_B = L_{(A+B)}; \quad \text{also } L_A - L_B = L_{(A-B)}.$$

That is: The sum or difference of two lunes is a lune whose angle is the sum or difference of the angles of the given lunes.

210. *A spherical triangle equals (in area) half the lune whose angle equals the spherical excess of the triangle.*

Hyp. Let E = the excess of $\triangle ABC$ located on sphere O.

Con. $\triangle ABC = \frac{1}{2}L_E$.



Proof. Complete the circles of which $\overset{\frown}{AB}$, $\overset{\frown}{AC}$, and $\overset{\frown}{BC}$ are arcs. Draw the diameters AOA', BOB', and COC'.

$$\triangle ABC + \triangle ACB' = L_B. \qquad \text{Why?}$$

$$\triangle ABC + \triangle A'BC = L_A; \ \triangle ABC + \triangle AC'B = L_C.$$

But $\qquad \triangle AC'B = \triangle A'CB'. \qquad$ § 194.

Substituting $\triangle A'CB'$ for $\triangle AC'B$ and adding, we get

$$2\triangle ABC + L_{180°} = L_{(A+B+C)}.$$

$$\therefore \quad 2\triangle ABC = L_{(A+B+C)} - L_{180°}, \text{ or } L_E. \qquad \therefore \quad \triangle ABC = \tfrac{1}{2}L_E.$$

211. *If the radius of a sphere is r, and the excess of $\triangle ABC$ is E, the formula for the area, S, of $\triangle ABC$ is:*

$$S = \frac{\pi r^2 E}{180}.$$

Proof. $L_E = \dfrac{\pi r^2 E}{90}$ $\triangle ABC$ $\frac{1}{2} L_E$ $\triangle ABC = \dfrac{\pi r^2 E}{180}$.

212. *The area of any spherical polygon whose excess is E* also *is given by the formula* $S = \pi r^2 E/180$.

This is true because the excess of the polygon is the sum of the excesses of the triangles into which the polygon can be divided by drawing diagonals from one vertex to each of the others; and the area of the polygon is the sum of the areas of these triangles. The excess of the polygon equals the sum of its angles diminished by $(n - 2)180°$. (§ 193.)

1. Find the area of the spherical triangle whose angles measure 125°, 130°, and 105°, on the sphere with radius 10 in.

2. Find the area of the spherical triangle on the sphere with radius 12 inches if the angles measure 103°, 112°, and 125°.

3. The sides of a triangle on the sphere with radius 15 in. measure 40°, 65°, and 95°. Find the area of its polar triangle.

4. What part of the sphere is intercepted by a trihedral angle whose dihedral angles measure 85°, 55°, and 100°?

5. What part of the sphere is a tri-rectangular triangle?

6. Compare the area of a tri-rectangular triangle of a sphere with that of the plane triangle whose sides are the chords of the spherical triangle, if the radius of the sphere is 14 inches.

7. The radius of the earth is about 4000 miles. What is the area of the earth correct to the nearest million square miles?

8. What is the area of the lune bounded by two meridians when the angle of the lune is 1°?

9. Assume that $\triangle ABC$ on the surface of the earth has angles A and B each measuring 90° and angle C measuring 15°. What part of the surface of the earth is bounded by this triangle?

10. What is the area of the zone bounded by 30° N latitude and 30° S latitude in terms of the radius r?

11. The north temperate zone is bounded by 23.5° N and 47° N. Express its area in terms of the radius, r, of the earth.

12. How does the area of a sphere change when its radius is doubled?

213. Maps of the world are essential for navigators. Since it is impossible to flatten out a sphere, map makers represent parts of the surface of the earth by figures that correspond to them, point for point.

One way of representing a sphere on a cylinder, point for point, would be as follows. (*a*) If Fig. (*a*) be revolved about XZ: ZW_2XW_1 generates sphere O of Fig. (*b*); $ABCD$ generates cylinder DB; W_1W_2, the "equator" W_1W_2; R_1R_2 and S_1S_2, parallels of latitude; $S_1'S_2'$, the circle $S_1'S_2'$ on the cylinder.

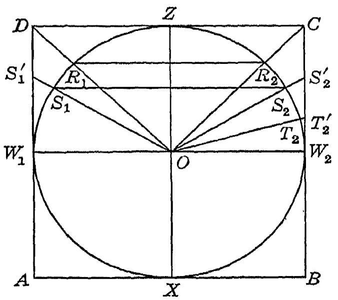


Fig. (*a*)

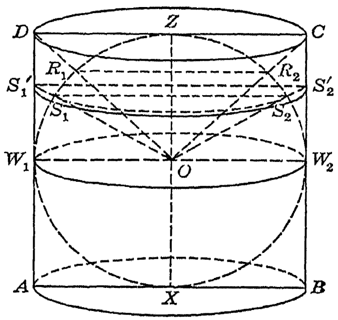


Fig. (*b*)

(*b*) Extend the radii of the sphere to intersect the cylinder. The projection of W_1 is W_1; of S_1 is S_1'; of R_2 is C; of $\odot S_1S_2$ is $\odot S_1'S_2'$; of arc W_1S_1 is *segment* W_1S_1'.

The parallels of latitude become circles on the cylinder, parallel to the equator, equal in length to it, but spaced at increasing distances as the distance from the equator increases.

The meridians become equidistant straight lines, perpendicular to the plane of the equator.

(*c*) If the cylinder is cut on line AD and flattened out, the surface of the sphere will be represented on a plane. The meridians would be represented by equally spaced straight lines perpendicular to the straight line that represents the equator; the parallels of latitude, by straight lines parallel to the equator, equal in length to the equator, and at distances that increase as the latitude increases. As the latitude increases, the surface represented is stretched laterally and, in the direction perpendicular to the equator, even more. In fact, for high latitudes, the elongation perpendicular to the equator produces extreme distortion.

214. The Mercator map of the world is suggested by but **is not a geometric projection** like that described in Section 213. The meridians on a Mercator map are represented by straight lines perpendicular to a straight line representing the equator and equally spaced, per degree of longitude. Any parallel of latitude is represented by a line parallel to the equator; the locations of the corresponding lines are determined so that, at any point on the map, distances parallel and perpendicular to the meridian through the point are stretched by the same per cent as compared to distances on the earth; hence, the angle formed by two intersecting paths on the earth is the *same as the angle formed by the representations of these paths on the map.*

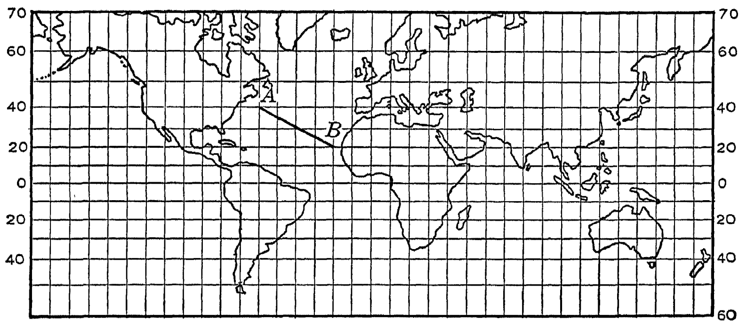
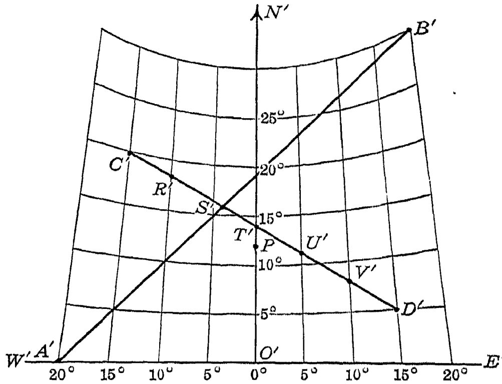


Fig. (*a*). Mercator projection of the world

On the map above, AB represents *a curve on the world which crosses meridians at a constant angle* because, on the map, AB crosses the meridians at a constant angle. Such a curve AB on the earth is called a **rhumb line;** that is, a rhumb line is a path with a *constant direction.* A rhumb line (in general) is *not* an arc of a great circle on the earth, because a great circle crosses meridians at a constant angle only when it is perpendicular to them. Therefore, the rhumb line path AB is not the *shortest* path from A to B. However, for short distances, the rhumb line path is not much greater than the great circle path from A to B. Navigators make Mercator charts for small sections of the globe. They compute the distances between parallels of latitude with the aid of a table of so-called **meridional parts.** On such a chart, the straight line between two points will represent the rhumb line path between the corresponding points on the world.

215. Gnomonic charts are used at sea to lay out the track between distant points. This track must be a great circle to be the shortest. *For convenience* the navigator wants a chart on which this *track appears as a straight line.*

For a part of the earth, such a chart is made by projecting that part from the center of the earth on a plane, T, tangent to the earth at a point near the center of the part being mapped; as P.



Radii to the equator lie in the plane of the equator. This plane intersects T in line $W'E'$.

Similarly the meridian through P, the point of tangency of T, is projected into $N'O'$, perpendicular to $W'E'$. North of the equator, the meridians are projected into straight lines converging at a point on $N'O'$. The distance between them on the equator increases east and west of $N'O'$.

The parallels of latitude are projected into the curved lines shown above the equator.

Every great circle of the earth will appear as a straight line on the chart of its vicinity, because the plane of such a great circle must intersect T in a straight line.

Thus: The great circle through A' (20° W, 0°) and B' (20° E, 30° N) must be the straight line $A'B'$ on the chart.

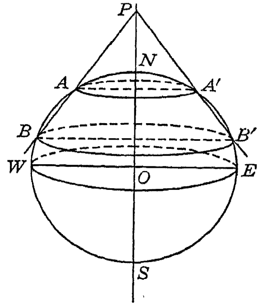
Conversely, every straight line on this chart corresponds to a great circle on the earth.

Thus: $C'D'$ must represent the great circle between C' (15° W, 20° N) and D' (15° E, 5° N).

Great $\odot CD$ will continually change its *direction* (§ 262).

Having laid out the track between C' and D', the navigator will lay out on Mercator charts the sections $C'R'$, $R'S'$, $S'T'$, etc. These sections, for short distances, will indicate the *rhumb line* tracks (§ 214) to be followed from point to point. The total track approximates the great circle track.

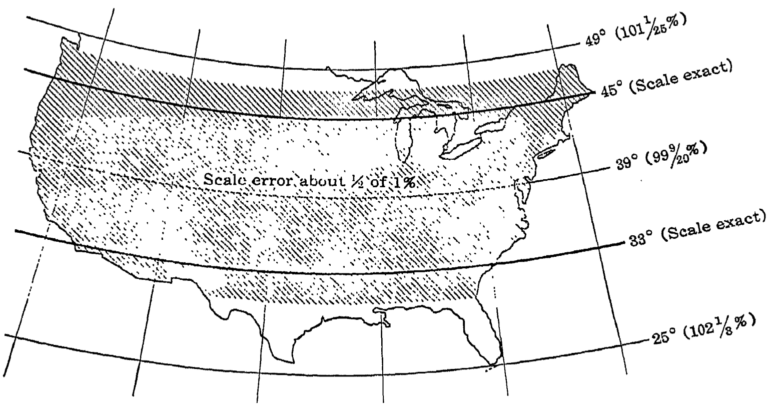
216. Lambert charts are designed to avoid distortion of shape as much as possible. Such charts are desirable for air navigation because pilots use landmarks from time to time.



Let AB be a chord of $\odot NWSE$. If the figure be revolved around PNS as axis, NWS generates the sphere O, A and B the small circles AA' and BB', W the equator, and PAB the *conical surface* of which P is the vertex and BB' a circular section.

If the sphere is projected on this conical surface *by drawing radii from center* O, the shapes on the sphere will not be changed greatly on the cone, *if* AA' *and* BB' *are selected with care.* Actually, the details of representing a part of the earth on such a conical surface are accomplished by use of mathematical formulas and not by a simple geometric projection.

The United States Coast and Geodetic Survey has selected for AA' the parallel 45° N, and for BB' 33° N. When the cone is cut along PAB and flattened, the map of the United States looks as follows.



Courtesy of the Civil Aeronautics Administration, Department of Commerce

Since the shapes on the map *conform* closely to actual shapes on the earth, this map is called a *conformal* representation of the earth.

For small parts of the surface of the earth distances and directions are nearly true. Therefore, the straight line between two points nearly equals the great circle path and indicates the distance and direction from one of the points to the other.

1. Is it possible to have a spherical triangle the sum of whose angles is:
(*a*) 180°? (*b*) 360°? (*c*) 600°?

2. A spherical triangle is equiangular.
(*a*) What is the upper limit of the size of each angle?
(*b*) What is the lower limit?

3. How large must the angle of a lune be in order that the lune shall equal one of the tri-rectangular triangles of the sphere?

4. Can a spherical quadrilateral have four of its angles right angles?

5. Do the equator, the meridians 30° *W* and 35° *W*, and the parallel of latitude 10° *N* form a spherical quadrilateral?

6. What part of the surface of a sphere is covered by the spherical triangle the sum of whose angles equals 420°?

7. What is the area of the sphere whose diameter is 20 in.?

8. What is the area of that part of the surface of the earth lying between the equator and the 60th and 70th meridians, West?

9. If the radius of a sphere is 15 in., find the area in square inches of the triangle whose angles measure:
(*a*) 45°, 75°, and 100°. (*b*) 100°, 120°, and 80°.
(*c*) 35°, 145°, and 55°. (*d*) 60°, 60°, and 100°.

10. What part of the surface of the sphere is covered by each of the triangles in Example 9?

11. What facts about the corresponding trihedral angle can you infer if a spherical triangle has:
(*a*) Three equal sides? (*b*) Two equal sides?
(*c*) Three equal angles? (*d*) Two equal angles?

12. A spherical triangle has two right angles. What can you infer about the corresponding trihedral angle?

13. Compare the area of a tri-rectangular triangle on the sphere of radius 15 in. with the area of the plane triangle whose sides are the *chords* of the sides of the spherical triangle.

14. A spherical angle intercepts an arc of 50° on the great circle whose pole is the vertex of the angle.
(*a*) How many degrees are there in the angle?
(*b*) How many degrees are there in the corresponding dihedral angle?

15. Two spheres have radii of 6 in. and 8 in. respectively.
(*a*) What is the ratio of the areas of their surfaces?
(*b*) What is the ratio of their volumes?

16. The angles of a triangle measure 75°, 80°, and 120°. What is the angle of a lune that equals this triangle?

17. What is the area of the zone between the equator and parallel of latitude 30° *N* on the earth?

Chapter XIV

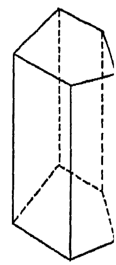
MEASUREMENT OF SOLIDS

217. (*a*) **The area** of a solid is the area of its surface.

(*b*) **The volume** of a solid is the volume of the space inclosed by the surface.

218. (*a*) **A polyhedron** is a closed surface consisting of parts of four or more planes; as the prisms and pyramids. The **faces** are the bounding plane surfaces; the **edges** are the intersections of the faces; the **vertices** are the intersections of the edges.

(*b*) **A prism** is a polyhedron that has two parallel faces, called the **bases,** with its remaining faces intersecting in parallel **lateral edges.** The most useful prisms are those in which the lateral edges are perpendicular to the bases.



(*c*) **A right prism** is a prism whose lateral edges are perpendicular to the bases.

(*d*) **The altitude** of a prism is the perpendicular between its bases.

219. It can be proved that:

(*a*) *The lateral edges of a prism are equal.*

(*b*) *The bases of a prism and any section parallel to them are congruent polygons.*

(*c*) *The lateral faces of a right prism are inclosed by rectangles.*

(*d*) *The lateral edges of a right prism equal the altitude.*

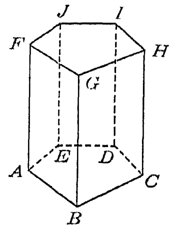
1. Prove that any two non-adjacent lateral edges of a prism determine a plane that is parallel to all the other lateral edges of the prism.

2. Prove that corresponding diagonals in the two bases of a prism are parallel and equal.

3. Prove that the section of a prism made by a plane that is parallel to any lateral edge is a parallelogram; and is a rectangle when the given prism is a right prism.

220. *The lateral area of a right prism equals the perimeter of its base multiplied by the length of its altitude.*

Let p = the perimeter of the base, and e = the length of a lateral edge, or of the altitude; let S = the lateral area, of right prism AI.



Area $ABGF = GB \times AB$, or $e \times AB$. Similarly $BCHG = e \times BC$; $CDIH = e \times CD$; etc.

Adding, and factoring out the edge e,

$$S = e(AB + BC + CD + \cdots \text{ etc.}) \text{ or}$$
$$S = ep.$$

1. Find the lateral area, S, of the right prism if:

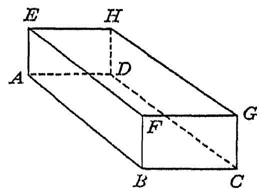
(a) $e = 9.25$ ft.; $p = 3$ ft. (b) $e = 8.5$ cm.; $p = 40$ cm.
(c) $e = 15'\,3''$; $p = 2'\,6''$. (d) $e = 3.5$ yd.; $p = 6$ ft.

2. What is the total surface area of the faces of 20 hexagonal concrete columns that are 10 ft. high, if the bases are inclosed by regular hexagons with 6 in. edges?

3. Using the formula $S = ep$, find:

(a) S when $e = 10.5$ cm. and $p = 8.5$ cm.
(b) p when $S = 145.85$ sq. in. and $e = 16.3$ in.
(c) e when $p = 6.5$ ft. and $S = 75$ sq. ft.

221. (a) **A parallelepiped** is a prism whose bases are inclosed by parallelograms.



(b) **A rectangular parallelepiped** is a right prism whose bases are inclosed by rectangles; as AG at the right.

(c) **A cube** is a rectangular parallelepiped whose base is inclosed by a square and whose edges equal its base edges.

222. It can be proved: *In a rectangular parallelepiped:*

(a) *Any two diagonals are equal.* (Prove $AG = EC$, etc.)

(b) *Any two diagonals bisect each other.*

(c) *The diagonals are concurrent.* The point of concurrency is called the **center** of the parallelepiped.

(d) *Any line through the center of a parallelepiped, and terminating in its faces, is bisected by the center.*

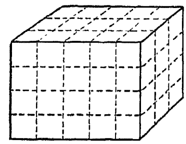
4. What is the length of the diagonal of a cube whose edge is 10 in. in length?

5. What figure is determined by two diagonally opposite edges of a cube?

223. (*a*) The **volume** of any solid is the number of units of space measure in the solid.

Thus: If the dimensions of a rectangular parallelepiped are 5, 3, and 4, the volume = $5 \times 3 \times 4$ space or cubic units.

P



This fact suggests the following.

(*b*) *The volume of any rectangular parallelepiped or rectangular solid is the product of its three dimensions, when the edges are measured by the same unit of length.*

If the dimensions are l, w, and h, then $V = \boldsymbol{lwh}$.

If the area of the base is B, then $V = \boldsymbol{hB}$.

(*c*) **Comparing rectangular solids.** Using the language of variation, the *volume of a rectangular solid varies jointly as its altitude and base;* so:

If its base is constant, the volume varies as its altitude;
If its altitude is constant, the volume varies as its base.

1. Find the weight of a steel plate that is 10′ long, 10″ wide, and $1\frac{1}{4}$″ thick, if the steel weighs .28 lb. per cubic inch.

2. How many packages $4'' \times 6'' \times 15''$ can be placed in a crate that has the dimensions 30″, 24″, and 30″?

3. How many cubic yards of concrete are needed for a platform for a gun if the platform must be 15′ square and 4′ deep?

4. Find the volume of a bar of steel 4 ft. long, 3 inches wide, and $\frac{1}{2}$ in. thick.

5. Find the weight of a steel plate 8 ft. long, 4 ft. wide, and $\frac{3}{4}$ in. thick, if the steel weighs .28 lb. per cu. in.

6. (*a*) A piece of steel is 10 ft. long and 1.5 ft. square. What is its weight?

(*b*) How wide a steel plate can be made from the piece of steel in part (*a*) if the length is still to be 10 ft. and the thickness is to be $\frac{3}{8}$ in.?

7. An irregular piece of steel is to be made from a certain pattern. The weight of the pattern is 48 pounds. What will the piece weigh, made of steel, if the wood used in the pattern weighs 28 lb. per cu. ft. and the steel weighs 490 lb. per cu. ft.?

8. What is the length of a diagonal of the rectangular parallelepiped whose edges measure 5″, 6″, and 4″, respectively?

9. If the three dimensions of a rectangular solid are doubled, what effect does that have:

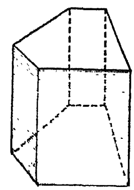
(*a*) On its total surface area?

(*b*) On its volume?

224. *The volume of any prism equals the product of the area of its base and the length of its altitude; or*

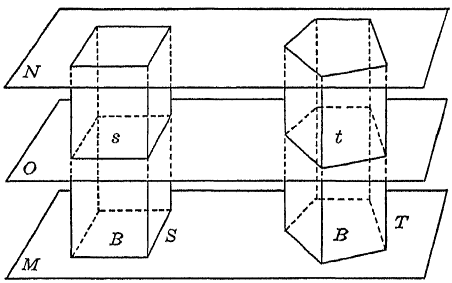
$$V = hB.$$

(*a*) *Intuitional verification* of this rule is suggested by the adjoining figure. The prism is pictured as consisting of a pile of sheets of paper, each congruent to the base. This suggests that the volume equals the area of the base multiplied by the height.



(*b*) An *interesting mathematical derivation* of the formula is based on a theorem that we shall assume and shall use several times in this chapter.

Cavalieri's Theorem. *If two solids lie between parallel planes, and if the sections of each made by every plane parallel to the parallel planes have the same area, then the two solids must have equal volumes.*



Thus: Assume that solids S and T lie between parallel planes M and N; also that any plane O, parallel to M and N, makes sections s and t of equal area; then the theorem states that $S = T$.

(*c*) If S and T are both prisms, then their bases must be equal in area, and their altitudes must also be equal.

Also, the sections s and t must be congruent to the bases by § 219(*b*). Therefore s does *equal* t.

If S is a rectangular solid, then $V = hB$.

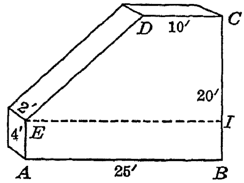
Therefore the volume of any prism is $V = hB$.

1. Find the volume of the right prism with altitude 18 in. if the base is inclosed by:

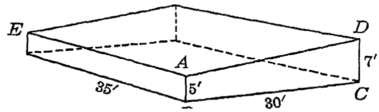
(*a*) A square having 4-in. sides.
(*b*) An equilateral triangle having 3-in. sides.
(*c*) A regular hexagon having 4-in. sides.

1. How many cubic yards of concrete are needed for a wall having the shape and dimensions in the adjoining figure?

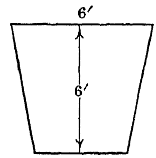
Suggestion. Consider *ABCDE* the base. Find the area by dividing it into a rectangle, *ABIE*, and a trapezoid, *EICD*.



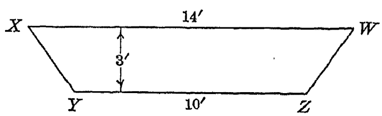
2. Find the volume of soil to be excavated to make a cellar on a hillside, having the shape and dimensions shown in the figure. Use the method suggested for Example 1.



3. The cross section of a trench has the shape and dimensions of the adjoining figure. How many cubic yards of earth must be moved to make such a trench, 100 yards in length?



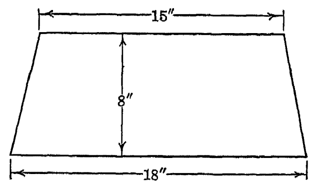
4. An oil storage tank in a vessel has a cross section with the shape and dimensions shown in this figure. If the tank is 15 ft. in length, how many gallons will the tank hold, since one cubic foot holds 7.5 gallons?



5. (*a*) What is the volume of a bar whose length is 4 ft. if the cross section is a trapezoid of which the upper and lower bases are 15″ and 18″ respectively, and the altitude is 8″?

(*b*) What is the weight of such a bar if it consists of:

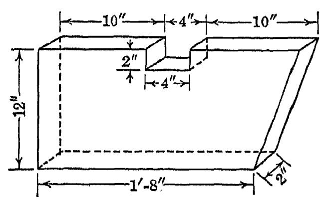
(1) Aluminum that weighs 165 lb. per cu. ft.?
(2) Copper that weighs 556 lb. per cu. ft.?



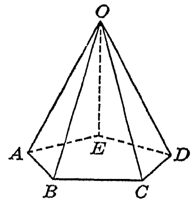
6. A steel plate has the shape and dimensions shown at the right.

(*a*) Find its volume.

(*b*) Find its weight if it is made of brass weighing 309 lb. per cu. ft.



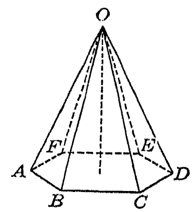
225. (*a*) **A pyramid** is a polyhedron (§ 218) that has three or more lateral faces, triangular in shape, that meet at a point, called the **vertex,** and, opposite this vertex, a **base** that intersects all the lateral faces.



The **edges** are the intersections of the *lateral faces.* The **altitude** is the perpendicular from the vertex to the base. The **lateral area** is the total area of the lateral faces.

(*b*) **A regular pyramid** has a base bounded by a regular polygon, and its vertex lying in the perpendicular to the base at its center.

Thus: O–ABCDEF is a regular pyramid.



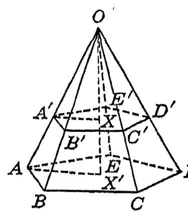
226. In a regular pyramid:

(*a*) *The lateral edges are equal.*

(*b*) *The lateral edges make equal angles with the altitude and with the base.*

(*c*) *The lateral faces are inclosed by congruent isosceles triangles.*

(*d*) *The* **slant height** *is the altitude of any of the lateral faces.*

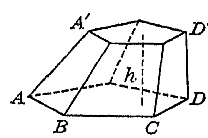


(*e*) *If a plane intersects all the lateral edges and is parallel to the base:*

(1) *It divides the lateral edges and altitude proportionally.*

(2) *The intersection is similar to the base and its area is to the area of the base as the square of its distance from the vertex is to the square of the altitude.*

227. A frustum of a pyramid is bounded by a plane, parallel to the base of the pyramid, and that part of the pyramid between the plane and the base; as frustum $A'D$.



$A'D'$ is the **upper base;** AD, the **lower.** The perpendicular between the bases is the **altitude.** The **slant height** of a frustum of a regular pyramid is the perpendicular between the edges of the bases on any face; as HH' in the figure of § 228.

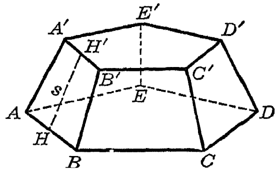
1. If a plane, parallel to the base of a pyramid, bisects the lateral edges, the perimeter of the section is one half the perimeter of the base.

228. The following facts can be proved:

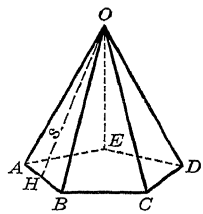
(*a*) *The upper base of any frustum is similar to the lower.*

(*b*) *The lateral faces of any frustum are inclosed by trapezoids.*

(*c*) *The lateral faces of a frustum of a regular pyramid are inclosed by congruent isosceles trapezoids. Therefore the lateral edges are equal, and make equal angles with the lower and upper base edges.*



229. *The lateral area of a regular pyramid equals one half the perimeter of its base times its slant height.*



Thus: If $O\text{–}ABCDE$ is a regular pyramid having slant height s, and p as the perimeter of its base, its lateral area, S, is given by the formula $S = \frac{1}{2}sp$.

Proof. Area $\triangle OAB = \frac{1}{2}s \times AB$. Since the faces are congruent,

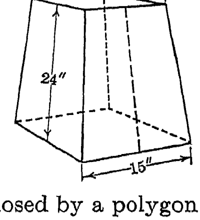
$$S = 5(\tfrac{1}{2}s \times AB) \quad \text{or} \quad \tfrac{1}{2}s(5AB).$$

But $p = 5AB$. $\quad \therefore \quad S = \frac{1}{2}sp.$

230. *The lateral area of a frustum of a regular pyramid equals one half its slant height times the sum of the perimeters of its bases.*

Thus: If p' is the perimeter of the upper base and p of the lower base, and s the slant height of the frustum $A'D$ of a regular pyramid, then $S = \frac{1}{2}s(p + p')$.

Proof. In the figure for § 228, the area of trapezoid $A'ABB'$ is $\frac{1}{2}s(A'B' + AB)$. Since the faces are congruent, then $S = 5(\frac{1}{2}s)(A'B' + AB)$.
$\therefore \quad S = \frac{1}{2}s(5A'B' + 5AB). \qquad \therefore \quad S = \frac{1}{2}s(p' + p).$



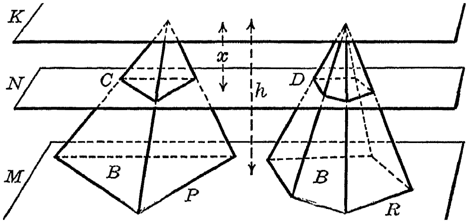
1. Some concrete pier blocks have the form of a frustum of a square pyramid, with dimensions as shown in the adjoining figure.

What is the lateral area of each block?

2. In Example 1, following § 227, what is the ratio of the area of the section to the area of the base?

3. The altitude of a pyramid is 24 in. Its base is inclosed by a polygon whose area is 40 sq. in. What is the area of the section of this pyramid made by a plane parallel to the base if the distance of it from the vertex is: (*a*) 6 in.? (*b*) 8 in.? (*c*) 15 in.?

231. *Pyramids having equal bases and altitudes are equal.*



Proof. Let pyramids P and R stand on plane M, and have their vertices in parallel plane K, so that they have equal altitudes. Let them have equal bases B. Let N be any plane, parallel to planes M and K, intersecting P in C and R in D.

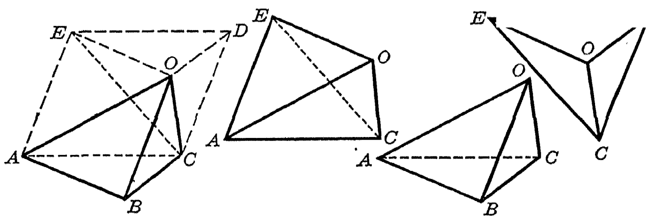
$C \sim B$; and $D \sim B$. § 226(e,2).

$C : B = x^2 : h^2$; also $D : B = x^2 : h^2$. § 226(e,2).

$C : B = D : B$, and then $C = D$. Why?

$\therefore \quad P = R$. § 224(b).

232. *The volume of a triangular pyramid equals one third the product of its altitude and base.*



Proof. Let h be the altitude from O to base ABC of pyramid O–ABC.

Consider prism EOD–ABC, with lateral edges parallel to OB, and base EOD parallel to ABC. Pass plane EOC through it. This plane and OAC separate the prism into the three triangular pyramids shown in the figure.

In O–ABC and C–EOD:

The bases are equal, since $\triangle ABC = \triangle EOD$.

The altitudes are equal, being the perpendicular between parallel planes ABC and EOD.

$\therefore \quad O\text{–}ABC = C\text{–}EOD$. § 231.

O–ABC and O–AEC may be viewed as C–OAB and C–EOA.

$C\text{-}OAB$ and $C\text{-}OEA$ have the same altitude; namely, the perpendicular from C to plane $ABOE$.

They have equal bases because $\triangle ABO = \triangle AEO$.

(These triangles are equal, because they have equal bases AB and EO, and the same altitude, the distance between parallels AB and EO.)

$\therefore \quad C\text{-}OAB = C\text{-}EOA.$ Why?

$\therefore \quad C\text{-}OAB = C\text{-}EOA = C\text{-}EOD.$

$\therefore \quad C\text{-}OAB$, or $O\text{-}ABC = \frac{1}{3}$ prism $EOD\text{-}ABC$.

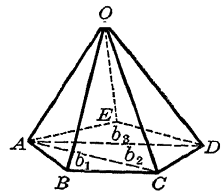
But the volume of the prism equals hB. § 224.

$\therefore \quad O\text{-}ABC = \frac{1}{3}hB.$ Why?

233. *The volume of any pyramid equals one third its altitude times its base.*

Proof. Pyramid $O\text{-}ABCDE$ can be divided by planes AOC, AOD into pyramids $O\text{-}ABC$, $O\text{-}ACD$, and $O\text{-}ADE$.

If h is the altitude of the pyramids and b_1, b_2, and b_3 are the bases of $O\text{-}ABC$, $O\text{-}ACD$, and $O\text{-}ADE$:



then $\quad O\text{-}ABC = \frac{1}{3}hb_1; \quad O\text{-}ACD = \frac{1}{3}hb_2;$

and $\quad O\text{-}ADE = \frac{1}{3}hb_3.$

$\therefore \quad P = \frac{1}{3}h(b_1 + b_2 + b_3).$

$\therefore \quad O\text{-}ABCDE = \frac{1}{3}hB.$

1. What is the volume of a square pyramid whose base edges measure 6 inches, and whose altitude is 20 inches?

2. Find the volume of the regular hexagonal pyramid whose base edges measure 8 in. and whose altitude is 24 in.

3. The center of a cube is joined to each vertex. Compare the volume of the pyramid that has as base one face of the cube and vertex at the center, with the volume of the cube.

4. A regular square pyramid has 12-in. lateral edges, with its base inscribed in a circle with 3-in. radius. Find the lateral area and the volume of the pyramid.

5. In the adjoining figure of a regular hexagonal pyramid, $VG \perp AB$ and $VO \perp$ base $ABCDEF$.

(*a*) Prove $OG \perp AB$.

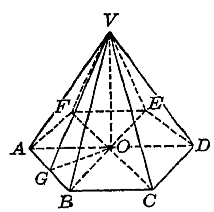
(*b*) If $VA = 16$ in. and $AB = 6$ in., find VG.

(*c*) Find the lateral area of the pyramid.

(*d*) Find the altitude VO.

(*e*) Find the volume of the pyramid.

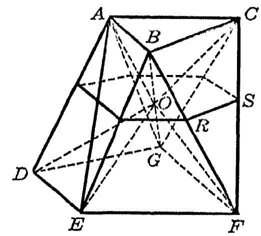
(*f*) Find the lateral area of the frustum made by the plane, parallel to the base, bisecting VO.



234. The formula for the volume of a frustum of a pyramid is $V = \frac{1}{3}h(B + b + \sqrt{Bb})$ where B is the lower base, b the upper base, and h is the altitude.

The development of this formula is rather involved, and is no longer considered part of the requirements in solid geometry. We shall accept it without proof.

235. A prismatoid is an interesting solid that includes as special cases most of the solids studied so far.



It has two parallel bases, such as ABC and $DEFG$, and lateral faces that are bounded by triangles, trapezoids, or parallelograms.

The formula for the volume is

$$V = \frac{1}{6}h(B + b + 4m)$$

where B = lower base, b = upper base, m = a mid-section, parallel to the bases, and h = the altitude.

NOTE. The proof of this formula is not difficult, but it is long. Planes are passed through O, any point on the mid-section, to divide the prismatoid into triangular pyramids, as in § 232.

1. From the formula for the prismatoid, derive that for the volume of:

(*a*) A prism. (*b*) A pyramid. (*c*) A frustum.

HINT. In (*a*) $b = B = m$.

2. Find the volume of a frustum of a square pyramid whose base edges measure 8″ and 4″ respectively, and whose altitude is 10″.

3. Find, correct to tenths, the volume of the frustum of a regular hexagonal pyramid whose lower base edges measure 10″ and whose altitude is 24″, if the upper base of the frustum is made by a plane 12″ from the vertex of the pyramid.

4. A monument is in the form of a frustum of a regular square pyramid 10 ft. high, the sides of whose bases measure 3 ft. and 2 ft. respectively, surmounted by a regular square pyramid each side of whose base is 2 ft. and whose altitude also is 2 ft.

What is the total weight of this monument if it is made of a stone that weighs 180 lb. per cubic foot?

5. If a plane is passed parallel to the base of a pyramid, bisecting the altitude:

(*a*) Express the area of the section, in terms of the base.

(*b*) Express the volume of the frustum below this plane in terms of the area of the lower base and the altitude.

1. What is the lateral area of the regular pyramid whose base is inclosed by a pentagon having 5-in. sides, and whose slant height is 15 in.?

2. The altitude of a regular hexagonal pyramid is 20 in.; the sides of the base measure 4 in.

(*a*) Find the slant height of this pyramid.

(*b*) Find its lateral area.

(*c*) Find its volume, correct to the nearest tenth of a cubic inch.

3. (*a*) In Example 1, what is the slant height of the frustum made by the plane that bisects the altitude?

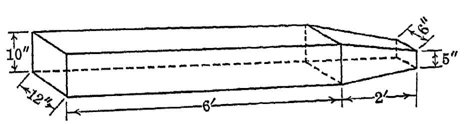
(*b*) What is the perimeter of the upper base of this frustum?

(*c*) What is the lateral area of the frustum?

4. (*a*) Do the parts of Example 3, starting from the pyramid described in Example 2.

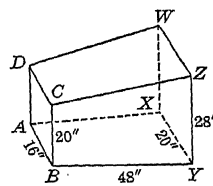
(*b*) Also find the volume of the frustum you get in this example.

5. A tinsmith is asked for the price to make a duct for a warm-air heating plant. The shape and dimensions of the duct are shown in the adjoining figure.



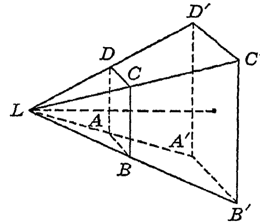
Determine the number of square feet of sheet metal required, with the understanding that this duct is open at both ends.

6. The adjoining figure represents a coal chute, to carry coal from a window in a basement to the hopper of a coal stoker. *ABCD* and *XYZW* are parallel rectangles 4 ft. apart. *BY* and *AX* are perpendicular to planes *ABCD* and *XYZW*. What is the cubical capacity of this coal chute?



Suggestion. Draw the mid-section, parallel to *ABCD*, and find its area. Then use § 235.

7. If *L* represents a light and *ABCD* a cardboard square between *L* and a wall parallel to *ABCD*, you see that the geometrical relations of the light, the cardboard and the shadow on the wall are indicated by the adjoining figure.



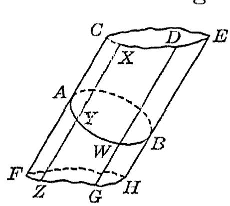
Suppose that *ABCD* is a 1-ft. square. Find the area of $A'B'C'D'$:

(*a*) When *ABCD* is halfway between *L* and $A'B'C'D'$.

(*b*) When *ABCD* is one third the way to the wall.

(*c*) When *ABCD* is one fourth the way to the wall.

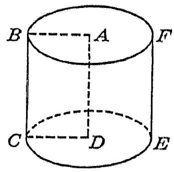
236. (*a*) **A closed cylindrical surface** is *generated* when a straight line XZ, the **generatrix,** moves parallel to an original position, with one of its points, Y, following a closed curved line, the **directrix,** $AYWB$.



Each position of the generatrix is an **element** of the surface.

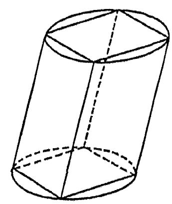
It can be proved that *parallel sections of a cylindrical surface are congruent.*

(*b*) **A cylinder** is formed when two parallel planes cut all the elements of a cylindrical surface. It consists of the parallel sections, the **bases** of the cylinder, and that part of the cylindrical surface between the bases.



(*c*) **A right circular cylinder** is formed when the bases are perpendicular to the elements and are inclosed by circles. The **axis** of the right circular cylinder is the line joining the centers of the bases. This axis is also the **altitude** of the right circular cylinder.

237. A prism is inscribed in a cylinder if its bases are inscribed in the bases of the cylinder, and its lateral edges are elements of the cylinder.



If we inscribe a square prism in a right circular cylinder, and then form a sequence of right prisms by successively doubling the number of faces, always keeping the boundary of the base a regular polygon, we ultimately get inscribed prisms that differ very little from the cylinder. This suggests the following postulate:

Postulate. *Any theorem about prisms that does not depend on some specific number of faces of the prism is equally true for a cylinder.*

238. The following theorems are consequences of § 237.

(*a*) *Parallel sections of a circular cylinder are congruent; if parallel to the base, they are circles.*

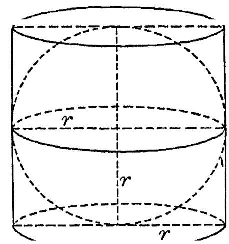
(*b*) *The lateral area of a right circular cylinder is the circumference of the base multiplied by the length of the altitude.* The formula is

$$S = 2\pi rh.$$

(*c*) *The volume of a right circular cylinder is the area of the base multiplied by the length of the altitude.* The formula is

$$V = \pi r^2 h.$$

1. *Prove.* The total area of a right circular cylinder having radius r and altitude h is $T = 2\pi r(r + h)$.



2. A right circular cylinder is circumscribed about a sphere, as shown in the adjoining figure.

(*a*) How does the radius of the cylinder compare with that of the sphere? The altitude with the diameter of the sphere?

(*b*) Compare the area of the sphere and the lateral area of the cylinder if the radius of the sphere is r.

3. A cylindrical tank car is 6 ft. in diameter and 25 ft. in length. How many gallons of fuel oil will it hold if one cubic foot holds 7.5 gallons?

4. How much brass is there in four feet of cylindrical brass tubing of which the outside diameter is $1\frac{1}{4}$ in. if the thickness of the metal is .06 in.?

5. The distance a piston moves from its lowest position in a cylinder to its highest is called its *stroke*. What is the amount of air or gas *displaced* by one stroke of the piston when the diameter of the piston is $3\frac{1}{2}$ in. and the stroke is 6 in.?

6. What is the lateral area and the volume of a right circular cylinder whose altitude is 15 in. and whose base has a 4-in. radius?

7. How many square feet of galvanized iron are needed to make 12 sections of hot-air furnace pipe if each section is 12 in. in diameter and 8 ft. long?

8. How many cubic feet of concrete are needed for a cylindrical well that is 6 ft. in inside diameter, 6 in. thick, and 12 ft. deep, inside measure. Include the base, of course.

9. What should the radius of a barrel be to hold 50 gallons, if the height of the barrel is 50 in.? Make use of the fact that one cubic foot will hold 7.5 gallons.

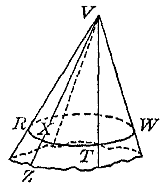
10. How many square feet of radiating surface, correct to hundredths, are there on a cylindrical heating pipe that is 2 in. in diameter and 18 ft. long?

11. How many square feet of steel are needed to make a cylindrical tank 10 ft. in diameter and 30 ft. high, allowing 5% extra for making laps?

12. (*a*) What is the net volume of a steel plate, $\frac{5}{8}''$ thick, having the shape of a trapezoid with upper base $15''$, lower base $24''$, and altitude $42''$, if a circular hole $5''$ in diameter is cut from the center of this plate?

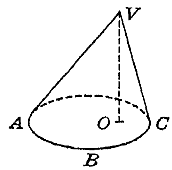
(*b*) What will the net weight be if the plate is made of steel weighing 490 lb. per cubic foot?

239. (*a*) **A closed conical surface** is generated when a straight line VZ, the **generatrix,** moves so that V remains at a fixed point and a point X of VZ follows a closed curve in a plane on which V does not lie. This curve is the **directrix,** RTW.

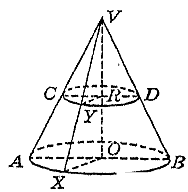


Each position of VZ is an element of the surface; V is the **vertex** or apex.

(*b*) **A cone** is formed when a plane cuts all the elements of a conical surface, on one side of the vertex; as cone $V\text{–}ABC$. A cone incloses a part of space, its **interior.** The **altitude** of a cone is the perpendicular from the vertex to the **base;** as VO.



(*c*) **A right circular cone** is formed when the base is inclosed by a circle, and the vertex lies in the perpendicular to the base at its center. This perpendicular is the **axis** and also the altitude; as VRO.

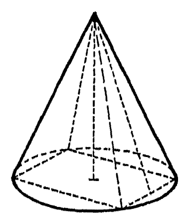


The slant height of a *right circular cone* is the line from the vertex to any point of the circle that bounds the base; as VYX.

(*d*) **A frustum** of a cone is formed when a plane, parallel to the base, cuts all the elements between the vertex and the base; as frustum CB in the figure opposite part (*c*).

Its **bases** are bounded by circles CYD and AXB; its **altitude** is RO, the perpendicular between its bases; the **slant height** of a frustum of a right circular cone is the part of any element of the cone between the bases; as XY.

240. If a square is inscribed in the base of a circular cone, and elements are drawn to the vertex, these elements are also the lateral edges of a square **pyramid inscribed in the cone.** The base of the pyramid is inscribed in the base of the cone; the altitude of the pyramid is the same as that of the cone; if it is a right circular cone, the pyramid is a regular pyramid. Similar statements may be made about the frustum of the pyramid inscribed in the frustum of the cone.



If we form a sequence of such pyramids by successively doubling the number of faces, keeping the base a regular polygon, we ultimately get inscribed pyramids that differ very little from the cone.

(*a*) **Postulate.** *Any theorem about pyramids that is independent of the number of faces of the pyramid is true also for a cone.* In particular:

(*b*) *A section of a cone made by a plane parallel to the base is similar to the base and its area is to the area of the base as the square of its distance from the vertex is to the square of the altitude.*

(*c*) *The lateral areas of a right circular cone and of a frustum of such a cone, respectively, are given by the formulas*

$$S = \pi RL \qquad \text{and} \qquad S = \pi L(R + r),$$

where L is the slant height, R is the radius of the lower base and r is that of the upper base of the frustum.

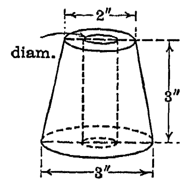
(*d*) *The volumes of a circular cone and of a frustum of such a cone, respectively, are given by the formulas*

$$V = \tfrac{1}{3}\pi R^2 h \qquad \text{and} \qquad V = \tfrac{1}{3}\pi h(r^2 + R^2 + rR).$$

1. What is the volume of the right circular cone whose altitude is 15 in. and whose radius is 4 in.?

2. Find the volume of the frustum of the right circular cone whose altitude is 15 in. and the radii of whose bases are 8 in. and 10 in. respectively.

3. Find the amount of steel in the bearing represented in the figure at the right. Observe that it has the form of a frustum of a right circular cone, with a cylindrical core running through its center. The necessary dimensions are given. Find the result correct to the nearest tenth of a cubic inch.



4. How many square yards of canvas are required for a tent consisting of a right circular cylinder, of altitude 10 ft. and diameter 80 ft., surmounted by a right circular cone that reaches 10 ft. above the side wall?

5. (*a*) Find the slant height of a right circular cone whose elements make 30° angles with the axis of the cone, when the radius of the base is 5 in.

(*b*) Find the lateral area of this cone.

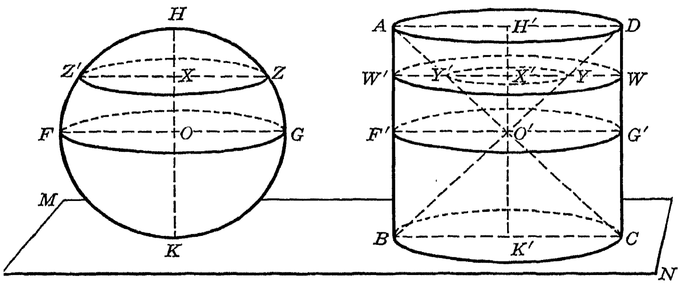
(*c*) Find the volume of this cone.

6. The altitude of a right circular cone is 25 in. and the diameter of its base is 10 in.

(*a*) What is the volume of the cone?

(*b*) What is the volume of the frustum formed when a plane is passed through the cone, parallel to the base and 15 in. from the vertex?

241. An interesting development of the formula for the volume of a sphere is that based on Cavalieri's Theorem.



Proof. Let sphere O and right circular cylinder AC stand on plane MN; let the radius of AC and of the sphere be r, and the altitude of AC be $2r$. From O', the center of the rectangle $ABCD$, let $O'A$ generate the circular cone $O'\text{–}AD$ and $O'B$ generate the circular cone $O'\text{–}BC$.

Through any point X of HK pass a plane parallel to MN, cutting the sphere in circle $Z'Z$, the cylinder in circle $W'W$, and the upper cone in circle $Y'Y$.

$$\text{Area of } \odot Z'Z = \pi(XZ)^2, \text{ or } \pi[r^2 - (OX)^2].$$
$$\text{Area of } \odot W'W = \pi(r)^2.$$
$$\text{Area of } \odot Y'Y = \pi(X'Y')^2, \text{ or } \pi(O'X')^2, \text{ or } \pi(OX)^2.$$
$$\therefore \quad \text{area of } (\odot W'W - \odot Y'Y) = \pi[r^2 - (OX)^2].$$
$$\therefore \quad \text{area of } (\odot W'W - \odot Y'Y) = \text{area of } \odot Z'Z.$$
$$\therefore \quad \text{vol. of sphere} = \text{vol. of (cylinder} - \text{cones)}. \qquad \S\ 224\ (b).$$
$$\therefore \quad \text{vol. of sphere} = 2r(\pi r^2) - 2(\tfrac{1}{3})(\pi r^2)(r)$$
$$= 2\pi r^3 - \tfrac{2}{3}\pi r^3.$$
$$\therefore \quad V = \tfrac{4}{3}\pi r^3.$$

1. What is the volume of the sphere having 10″ radius?

2. What is the weight of a spherical ball $\frac{1}{2}$-in. thick if its outside diameter is 8 in. and if one cubic inch of the steel weighs .28 lb.?

3. A storage tank for gasoline is spherical in form, having diameter 20 ft. in length. How many gallons of gasoline does the tank hold if one cubic foot holds 7.5 gal.?

4. A cylindrical vessel 10 in. in diameter was filled to the top with water. Then a spherical ball was immersed in it. Naturally some of the water ran out of the vessel. After the ball was removed, it was found that the surface of the water had dropped 2.5 in. below the top of the vessel. What was the diameter of the ball?

1. A pyramid, with altitude 12 in. and base area of 30 sq. in., is cut by a plane parallel to the base that bisects the altitude.

(*a*) What is the area of the section parallel to the base?

Suggestion. Use § 226(*e*).

(*b*) What is the volume of the original pyramid?

(*c*) What is the volume of the part above the plane?

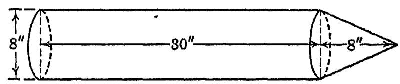
(*d*) What is the volume of the part below the plane?

2. A right prism with an irregular base has lateral edges that are 15 in. in length. The edges of the base measure 5, 6, 8, and 4 inches respectively. What is the lateral area of this prism?

3. The altitude of a right circular cone is 12 in., and the radius of its base is 4 in. Find its volume and lateral area.

4. A right circular cone with radius 6″ and altitude 15″ is cut by a plane parallel to its base, that bisects its altitude. Find the lateral area and volume of the part of the original cone above the intersecting plane.

5. A bombshell has, approximately, the shape and dimensions shown in the adjoining figure. Find:



(*a*) The total area of its exterior surface, correct to tenths.

(*b*) Its approximate volume also correct to tenths.

6. A sphere has radius of 10 in. It is cut by a plane 3 in. from its center. What is the radius of the small circle that is the intersection? (Correct to tenths.)

7. What is the area of the zone of one base that lies between the small circle obtained in Example 6 and the pole of it that is nearest to it?

8. A circle is inscribed in a square $ABCD$, tangent to its sides at X, Y, Z, and W, respectively. The diagonals of the square intersect at O, and diameter XZ passes through O.

(*a*) If the whole figure is revolved around XZ as axis, what solid is generated by:

(1) Circle $XYZW$? (2) $ABCD$? (3) AC?

(*b*) Find the volume of each of the solids generated in part (*a*) if the side of the original square is $2s$.

(*c*) Compare the volume generated by $XYZW$ with the difference in volumes of the solids generated by $ABCD$ and AC, respectively.

9. In Example 8 find the relation between the lateral areas of the solids generated by $XYZW$ and $ABCD$.

10. A sphere has radius of 12 in. It is cut by two parallel planes, one 3 in. from the center and the other 6 in. from the center. Find the area of the zone bounded by the resulting small circles.

APPENDIX FOR PART II

A PARTIAL SYLLABUS OF PLANE GEOMETRY

1. A segment (of a line) is the part of a straight line consisting of two points of the line and the part of the line between them.

2. A ray is that part of a straight line consisting of a point of it and all the line on one side of that point.

3. An angle consists of two rays that have a common end point.

4. A simple closed line is a line such that a point can move from any initial position around the line back to its initial position without passing through any intervening point more than once.

5. A circle is a simple closed curve whose points are all equidistant from a point called the center.

6. (*a*) Two angles are complementary when their sum is 90°.

(*b*) Two angles are supplementary when their sum is 180°.

7. Vertical angles are equal.

8. If two adjacent angles have their exterior sides in a straight line, they are supplementary; and conversely.

9. The total angular magnitude around a point is 360°.

10. Triangles are congruent when they can be made to coincide.

11. Two triangles are congruent:

(*a*) If two sides and the included angle of one are equal to two sides and the included angle of the other.

(*b*) If two angles and the included side of one, etc.

(*c*) If the three sides of one, etc.

(*d*) If two angles and any side of one, etc.

(*e*) If they are right triangles and have the hypotenuse and a leg of one, etc.

12. Two points equidistant from the ends of a segment determine the perpendicular-bisector of the segment.

13. One and only one perpendicular can be drawn to a line:

(*a*) At a point of the line.

(*b*) From a point not on the line.

14. Two straight lines are parallel when they lie in the same plane and do not meet however far extended.

15. There is only one parallel to a line through a point not on the line.

16. If two lines are perpendicular to the same line, they are parallel.

17. If a line is perpendicular to one of two parallels, it is perpendicular to the other one also.

18. The sum of the angles of a triangle equals 180°.

19. The sum of the angles of a polygon having n sides is $(n-2)180°$.

20. Only one angle of a triangle can be a right angle.

21. Any point in the perpendicular-bisector of a segment is equidistant from the ends of the segment; and conversely.

22. Any point on the bisector of an angle is equidistant from the sides of the angle; and conversely.

23. A parallelogram is a quadrilateral whose opposite sides are parallel; its opposite sides can be proved equal.

24. A quadrilateral is a parallelogram:

(*a*) If two sides are equal and parallel.

(*b*) If the opposite sides are equal.

25. If two sides of a triangle are equal, the opposite angles are equal; and conversely.

26. A trapezoid is a quadrilateral that has just two parallel sides.

27. If three or more parallels cut off equal segments on one transversal, they do likewise on every transversal.

28. The axioms of inequality.

Ax. 9. If equals be added to unequals, the sums are unequal in the same order.

Ax. 10. If equals be subtracted from unequals, the remainders are unequal in the same order.

Ax. 11. If $a > b$, and $b > c$, then $a > c$.

Ax. 12. If unequals be added to unequals of the same order, the sums are unequal in the same order.

Ax. 13. If unequals be subtracted from equals or from unequals of opposite order, the remainders are unequal, and the order is opposite to that of the subtrahend.

29. If two sides of a triangle are unequal, the angles opposite are also in the same order; and conversely.

30. The perpendicular is the shortest segment from a point to a line.

31. If two triangles have two sides of one equal to two sides of the other and the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second; and conversely.

32. In the same circle or equal circles:

(*a*) If central angles are equal, their arcs and chords are equal; and conversely.

(*b*) If central angles are unequal, their arcs and chords are also; and conversely.

(*c*) If chords are equal, they are equidistant from the center of the circle; and conversely.

33. A line from the center of a circle perpendicular to a chord bisects the chord and its subtended arcs; and conversely.

34. If a line is perpendicular to a radius at its outer extremity, it is tangent to the circle; and conversely.

35. A central angle has the same measure as its arc.

36. An inscribed angle has the same measure as half its arc.

37. A regular polygon is equilateral and equiangular.

38. A line parallel to one side of a triangle divides the other two sides proportionally; and conversely.

39. Two polygons are similar if they are mutually equiangular and their corresponding sides are proportional; and conversely.

40. Two triangles are similar:

(*a*) If they are mutually equiangular.

(*b*) If an angle of one equals an angle of the other and the sides including these angles are proportional.

(*c*) If their corresponding sides are proportional.

41. The square of (or on) the hypotenuse of a right triangle equals the sum of the squares of the other two sides.

42. Polygons are equal when they have the same area.

43. The perimeters of similar polygons are to each other as any two corresponding sides; and the areas as the squares of any two corresponding sides.

PART III. SPHERICAL TRIGONOMETRY

Chapter XV

RIGHT SPHERICAL TRIANGLES

242. Introduction. If three points A, B, and C, on a sphere, are joined by arcs of great circles of the sphere, the figure ABC is called a *spherical triangle* whose sides are the arcs AB, BC, and AC and whose angles are the angles formed by the arcs. **Spherical trigonometry** is concerned with deriving and applying trigonometric formulas involving the sides and angles of a spherical triangle.

We recall that an arc of a great circle is measured by the angle subtended by the arc at the center of the sphere. Hence, the length of the arc can be found if we know its angular measure and the radius of the sphere. Therefore, both the *sides and angles* of any spherical triangle will be described in *angular* units.

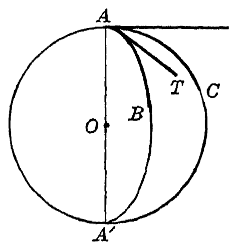


FIG. 1

The angle BAC formed by arcs AB and AC of great circles is measured by the angle TAM formed by the tangents AT and AM to these arcs at A. In Figure 1, AT and AM are perpendicular to AA'. Thus, the angle TAM also measures the *dihedral angle* formed by the planes $A'OAB$ and $A'OAC$ in which the arcs lie.

We agree to consider only spherical triangles each of whose sides and angles is less than 180°. As a standard notation for the spherical triangle ABC, we shall let α, β, and γ represent the angles at A, B, and C, respectively, and let a, b, and c be the corresponding opposite sides. The letter O will always represent the center of the sphere. Then, for any triangle ABC, we can construct a trihedral angle O–ABC, as in Figure 2, whose *dihedral angles* have the same measures as the *angles* α, β, and γ, and whose

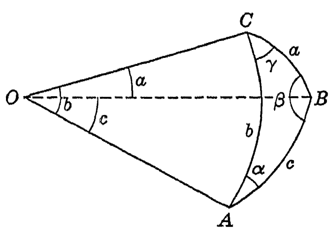


FIG. 2

face angles AOB, COB, and COA have the same measures as the *sides* of triangle ABC.

Illustration 1. In Figure 2, side CB lies in plane OBC and is measured by $\angle COB$; thus, in the figure we display a as a label for the measure of $\angle COB$ and *also* of arc CB. Similarly, $b = \angle COA$ and $c = \angle BOA$.

NOTE 1. If a spherical triangle ABC has a side or an angle greater than 180°, various arcs of the great circles which intersect to form triangle ABC also form related triangles each of whose sides and angles is less than 180°. It can be verified later that any of the typical problems concerning the given triangle may be solved by investigating some one of these related triangles. Hence, our agreement to consider only those triangles whose sides and angles are all less than 180° is not an essential restriction.

243. Formulas for right spherical triangles. A right spherical triangle is one which has at least one angle equal to 90°. In such a triangle ABC, we shall always letter the vertices so that $\gamma = 90°$. Then, we shall prove the following formulas.

(I)	$\sin a = \sin c \sin \alpha$.	(VI)	$\sin b = \sin c \sin \beta$.
(II)	$\tan b = \tan c \cos \alpha$.	(VII)	$\tan a = \tan c \cos \beta$.
(III)	$\tan a = \sin b \tan \alpha$.	(VIII)	$\tan b = \sin a \tan \beta$.
(IV)	$\cos \alpha = \sin \beta \cos a$.	(IX)	$\cos \beta = \sin \alpha \cos b$.
(V)	$\cos c = \cos a \cos b$.	(X)	$\cos c = \cot \alpha \cot \beta$.

Proof when $a < 90°$ and $b < 90°$. **1.** Through any point D on OB, in Figure 3, draw a plane perpendicular to OA, intersecting planes OAC, OBC, and OAB in EF, FD, and DE, respectively.

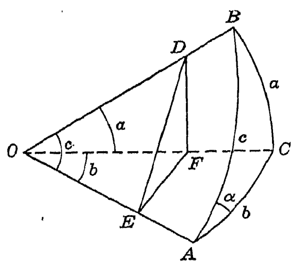


FIG. 3

2. $DE \perp OA$ and $EF \perp OA$ because OA is perpendicular to plane DEF. Hence, $\angle DEF$ measures the *dihedral angle* whose edge is OA; therefore, $\angle DEF = \alpha$. Also, $\triangle OEF$ and OED have right angles at E.

3. Plane DEF is perpendicular to plane OAC because plane DEF is perpendicular to OA which lies in OAC. Hence, DF is perpendicular to plane OAC because DF is the intersection of planes OBC and DEF which are perpendicular to plane OAC. Hence, $DF \perp OC$ and $DF \perp EF$, and $\triangle DOF$ and DEF have right angles at F.

4. From Figure 3, $\quad a = \angle FOD, \quad b = \angle FOE, \quad$ and $\quad c = \angle EOD.$

5. Take OD as a unit for measuring lengths; then $OD = 1$.

6. From right $\triangle ODF$ and ODE, since $OD = 1$,

$$\sin a = FD; \quad \cos a = OF; \quad \sin c = ED; \quad \cos c = OE. \tag{1}$$

7. From $\triangle OEF$, $OE = OF \cos b$; or, $\quad \cos c = \cos a \cos b$.

8. From $\triangle OEF$, $EF = OF \sin b$ and $EF = OE \tan b$. Hence,

$$EF = \cos a \sin b; \quad EF = \cos c \tan b.$$

9. Since $\angle DEF = \alpha$, $\tan \alpha = \dfrac{FD}{EF} = \dfrac{\sin a}{\cos a \sin b}$; or $\tan \alpha = \dfrac{\tan a}{\sin b}$.

This proves (III). Similarly, the student should prove (I) and (II) by use of $\sin \alpha$ and $\cos \alpha$ as obtained from $\triangle DEF$.

10. By symmetrical reasoning, where the places of a and α would be interchanged with the places of b and β, respectively, we would obtain (VI), (VII), and (VIII) in place of (I), (II), and (III).

11. From (II), (V), and (VI), we obtain (IV):

$$\cos \alpha = \frac{\tan b}{\tan c} = \frac{\sin b \cos c}{\cos b \sin c} = \frac{(\sin c \sin \beta)(\cos a \cos b)}{\sin c \cos b},$$

or

$$\cos \alpha = \sin \beta \cos a.$$

12. By interchanging α with β and therefore changing a to b in (IV) we obtain (IX). If we solve (IV) for $\cos a$ and (IX) for $\cos b$ and use (V) we obtain (X).

★NOTE 1. Recall that a right spherical triangle is said to be *birectangular* or *trirectangular* according as it has just two or just three angles equal to 90°. By construction on a sphere, the student can easily verify that a triangle is birectangular when and only when it has just two sides equal to 90°, and trirectangular when and only when all sides equal 90°. For such triangles, some of the preceding formulas become meaningless because $\tan 90° = \infty$, while from geometry it can be verified that the remaining formulas state evident truths.

★NOTE 2. From the preceding proof it can be seen that (I) to (X) will be true in *all* cases if (I), (II), (III), and (V) are *always true*. By use of Figure 4, a moderate modification of the preceding proof would establish (I), (II), (III), and (V) when ($a > 90°$, $b < 90°$).

Other figures would apply if we had ($a > 90°$, $b > 90°$) or if ($a < 90°$, $b > 90°$).

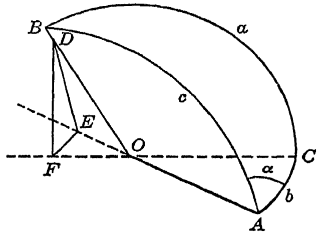


FIG. 4

244. Napier's rules. A convenient means for remembering (I) to (X) was formulated by Napier, the inventor of logarithms. To describe his method, we let the complements of α, β, and c be denoted by $Co\text{-}\alpha$, $Co\text{-}\beta$, and $Co\text{-}c$; that is, $Co\text{-}\alpha = 90° - \alpha$; etc. Then, we shall call (a, b, $Co\text{-}\alpha$, $Co\text{-}c$, $Co\text{-}\beta$) the five **circular parts** for the triangle ABC, and shall think of these parts arranged either around a *circle* or around the *triangle* ABC in the order (a, b, α, c, β). Thus, to each circular part there correspond two *adjacent* parts and two *opposite* parts. Then, if each of the following rules is applied to each of the circular parts, we obtain formulas (I) to (X).

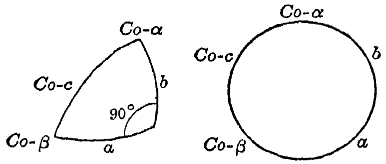


FIG. 5

Rule 1. *The sine of any circular part is equal to the product of the tangents of the two adjacent parts.*

Rule 2. *The sine of any circular part is equal to the product of the cosines of the two opposite parts.*

In applying these rules, recall that, for any angle θ, any function of $(90° - \theta)$ equals the *co*-function of θ.

Illustration 1. The parts *adjacent* to $Co\text{-}\beta$ are $Co\text{-}c$ and a; the parts *opposite* to $Co\text{-}\beta$ are $Co\text{-}\alpha$ and b. Hence, from Rules 1 and 2,

$$\sin(Co\text{-}\beta) = \tan(Co\text{-}c)\tan a; \quad \text{or, } \cos\beta = \cot c \tan a; \tag{1}$$
$$\sin(Co\text{-}\beta) = \cos(Co\text{-}\alpha)\cos b; \quad \text{or, } \cos\beta = \sin\alpha\cos b. \tag{2}$$

We verify that (1) is equivalent to (VII) and (2) is (IX). Rules 1 and 2 do not *prove* (I) to (X) but furnish a means for *remembering* them.

By Napier's rules, we can write a formula connecting any three of the parts (a, b, α, β, c) without referring to formulas (I) to (X).

Illustration 2. To write a formula connecting α, β, and a, we first notice that $Co\text{-}\beta$ and a are *opposite* to $Co\text{-}\alpha$. Hence, by use of Rule 2,

$$\sin(Co\text{-}\alpha) = \cos(Co\text{-}\beta)\cos a; \quad \text{or,} \quad \cos\alpha = \sin\beta\cos a.$$

245. Rules for species. We shall say that two angular magnitudes are of the *same* or of *different* **species** according as they terminate in the *same* or in *different* quadrants. In this statement the angles are thought of in their standard positions on a coordinate system as in the definition of the trigonometric functions.

Illustration 1. 130° and 75° are of *different* species.

The following theorems are useful in discussing the species of parts of a spherical triangle.

Theorem I. *The sum of two sides of a spherical triangle is greater than the third side.**

Theorem II. *If two sides of a spherical triangle are unequal, the angles opposite are unequal and the greater angle is opposite the greater side. If two sides are equal, the angles opposite are equal.**

Theorem III. *In a right spherical triangle ABC,*

1. *a and α are of the same species;*

2. *b and β are of the same species;*

3. *if $c < 90°$, then a and b are of the same species;*

4. *if $c > 90°$, then a and b are of different species.*

Proof. From (III) of Section 243, $\tan a = \sin b \tan \alpha$. Since $b < 180°$, $\sin b > 0$. Hence, $\tan a$ and $\tan \alpha$ are *both positive,* and a and α are both *less* than 90°, or $\tan a$ and $\tan \alpha$ are *both negative,* and hence a and α are both *between* 90° *and* 180°. Thus, a and α are of the same species. (Parts 2, 3, and 4 will be proved by the student in the next exercise.)

EXERCISE 1

1. Apply Napier's rules to each of the five circular parts and verify that (I) to (X) are thus obtained.

Without reference to formulas (I) *to* (X), *by use of Napier's rules obtain a formula connecting the given parts of the right spherical triangle ABC.*

2. (c, α, β). **4.** (b, c, β). **6.** (α, β, b). **8.** (a, c, α). **10.** (a, b, α).
3. (a, c, β). **5.** (a, b, c). **7.** (α, β, a). **9.** (b, c, α). **11.** (a, b, β).

12. Prove Parts 2, 3, and 4 of Theorem III by use of (V) and (VIII).

246. Solution of a right spherical triangle *ABC*. Suppose that values are assigned for any two parts of the triangle other than the right angle γ. Then, we shall find that there will exist *just one solution* for the unknown parts, or *just two solutions,* or *no solution,* depending on the nature of the data. Any solution which exists, or the conclusion that *no* solution exists, can be arrived at by use of formulas (I) to (X). An inspection of the possibilities as to two given

* Theorems I and II are proved in Part II of this text.

parts shows that two solutions cannot exist except when the given parts are (a *and* α) or (b *and* β); in either of these cases the data yield either two solutions or no solution.

In preparing the computing form for the solution of a right spherical triangle, we may employ Napier's rules and the methods of Problems 2 to 11 of Exercise 1 in writing the formulas to be used.

Example 1. Solve triangle ABC if $a = 32° 24'$ and $c = 49° 15'$.

Solution. To find any unknown part, we use that formula which involves this part and the given parts. Thus, to find b, we use (V).

Formulas	**Computation**
	Data: $a = \mathbf{32° 24'}$; $c = \mathbf{49° 15'}$.
(V) $\cos b = \dfrac{\cos c}{\cos a}$.	$\log\cos c = 9.8148 - 10$ (Table VI) $\log\cos a = 9.9265 - 10$ $(-)$ $\log\cos b = 9.8883 - 10$; $b = 39° 21'$.
(I) $\sin\alpha = \dfrac{\sin a}{\sin c}$.	$\log\sin a = 9.7290 - 10$ $\log\sin c = 9.8794 - 10$ $(-)$ $\log\sin\alpha = 9.8496 - 10$; hence,* $\alpha = 45° 1'$ or $\cancel{134° 59'}$.

NOTE. $134° 59'$ is ruled out by Part 1 of Theorem III, Section 245: since $a < 90°$, it follows that $\alpha < 90°$.

(VII) $\cos\beta = \tan a \cot c$. | The student should compute: $\beta = 56° 51'$.

Summary. $\alpha = 45° 1'$; $\beta = 56° 51'$; $b = 39° 21'$.

Check. Use (IX) because it involves all computed parts.

$\cos\beta = \cos b \sin\alpha$. $\quad \cos b = .7733$ (Table VII)

$(\times)\ \sin\alpha = .7073$

(Table VII) $\cos\beta = .5468 \rightarrow$ $\quad \cos b \sin\alpha = .5469.$ (By arithmetic)

Comment. If the check formula were tested by finding the *logarithms* of the sides, we would thus check the *logarithmic* part of the solution, but not the *values* of b, α, and β. However, the logarithmic check would have the advantage of being more quickly accomplished.†

In the logarithmic computation of a product or a quotient, the logarithm of the numerical value of any *negative factor* F will be labeled "**(n) log F.**" The logarithmic work will then give the *numerical value* of the product or quotient; its sign will be $+$ or $-$ according as an *even* or an *odd* number of factors are negative.

* Recall that to every value of $\log\sin\theta$, where $\sin\theta > 0$ and $\neq 1$, there correspond *two* values of θ, which are *supplementary* and between 0° and 180°.

† The instructor may desire to direct the use of logarithmic checking.

Example 2. Solve triangle ABC if $a = 63° 40'$ and $b = 117° 25'$.

Formulas	**Computation**
	Data: $a = \mathbf{63° 40'}$; $b = \mathbf{117° 25'}$.
(VIII) $\tan \beta = \dfrac{\tan b}{\sin a}$. $\tan 117° 25' = -\tan 62° 35'$.	(n) log tan b = 10.2850 − 10 log sin a = 9.9524 − 10 (−) (n) log tan β = 0.3326; from Table VI, log tan 65° 4′ = 0.3326; ∴ β = 114° 56′.
(V) $\cos c = \cos a \cos b$. $\cos 117° 25' = -\cos 62° 35'$.	log cos a = 9.6470 − 10 (n) log cos b = 9.6632 − 10 (+) (n) log cos c = 9.3102 − 10; from Table VI, log cos 78° 13′ = 9.3102 − 10; ∴ c = 101° 47′.
Formula (III).	The student may verify: $\alpha = 66° 17'$.
Summary.	$\alpha = 66° 17'$; $\beta = 114° 56'$; $c = 101° 47'$.

Check. Use $\cos c = \cot \alpha \cot \beta$, with logarithms.

Comment. In the preceding solution, $\tan \beta$ is negative and the logarithm of its numerical value is 0.3326. Therefore, β is in quadrant II and $\tan \beta$ is numerically equal to tan 65° 4′. Hence, $\beta = 180° - 65° 4' = 114° 56'$.

Example 3. Solve triangle ABC if $\alpha = 43° 21'$ and $a = 36° 42'$.

Formulas	**Computation**
	Data: $\alpha = \mathbf{43° 21'}$; $a = \mathbf{36° 42'}$.
(III) $\sin b = \dfrac{\tan a}{\tan \alpha}$.	log tan a = 9.8723 − 10 log tan α = 9.9750 − 10 (−) log sin b = 9.8973 − 10; hence, b = 52° 8′ or 127° 52′.
(I), to find c. **(IV), to find β.**	The student may verify the results: c = 60° 31′ or 119° 29′; β = 65° 7′ or 114° 53′.

Grouping. By Part 3 of Theorem III, Section 245, if $c = 60° 31'$ then $b = 52° 8'$, and therefore (by Part 2) $\beta = 65° 7'$. The other values form a second solution.

Summary. *First solution:* $b_1 = 52° 8'$; $\beta_1 = 65° 7'$; $c_1 = 60° 31'$.
Second solution: $b_2 = 127° 52'$; $\beta_2 = 114° 53'$; $c_2 = 119° 29'$.

Comment. In Figure 6, the first solution of Example 3 is represented by triangle ABC, and the second by triangle $A'BC$ (the figure is not true to scale, but essential features are appropriately represented).

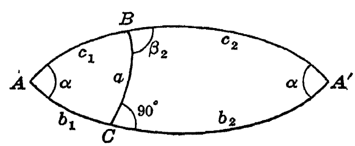


FIG. 6

EXERCISE 2

Solve the right spherical triangle ABC. If computation is necessary, use four-place or five-place logarithms, and check the solution.

1. $a = 45° 20'$; $c = 63° 10'$.
2. $b = 62° 10'$; $c = 75° 20'$.
3. $\alpha = 58° 20'$; $c = 85° 40'$.
4. $\beta = 73° 20'$; $c = 65° 30'$.
5. $a = 83° 45'$; $b = 45° 26'$.
6. $a = 61° 26'$; $b = 79° 32'$.
7. $b = 38° 15'$; $\beta = 52° 10'$.
8. $a = 43° 28'$; $\alpha = 57° 40'$.
9. $a = 61° 26'$; $b = 110° 30'$.
10. $a = 103° 15'$; $b = 58° 23'$.
11. $\alpha = 48° 26'$; $\beta = 100° 40'$.
12. $\alpha = 120° 15'$; $\beta = 70° 32'$.
13. $a = 63° 28'$; $\alpha = 48° 14'$.
14. $\alpha = 22° 10'$; $\beta = 37° 20'$.
15. $a = 127° 30'$; $\beta = 114° 27'$.
16. $b = 103° 18'$; $\beta = 135° 14'$.
17. $\alpha = 100° 15'$; $\beta = 160° 38'$.
18. $b = 67° 35'$; $\alpha = 108° 30'$.
19. $a = 114° 20'$; $\alpha = 96° 18'$.
20. $b = 126° 14'$; $\beta = 105° 30'$.
21. $b = 90°$; $a = 36°$.
22. $a = 90°$; $\beta = 90°$.
23. $a = 63°$; $\alpha = 128°$.
24. $c = 35° 25'$; $b = 63° 46'$.

Solve by use of five-place logarithms and check.

25. $\alpha = 65° 27.3'$; $\beta = 49° 18.6'$.
26. $a = 35° 41.7'$; $c = 73° 24.5'$.
27. $\alpha = 124° 14.8'$; $a = 147° 15.2'$.
28. $\beta = 108° 19.4'$; $b = 157° 13.2'$.
29. $c = 128° 31.3'$; $\alpha = 36° 14.9'$.
30. $b = 36° 45.2'$; $\alpha = 131° 14.7'$.

What can be said about the other parts of the spherical triangle ABC under the given condition? Then, check (I) *to* (X) *for the triangle.*

31. $\alpha = \gamma = 90°$.
32. $\gamma = a = 90°$.
33. $\gamma = c = 90°$.
34. $\gamma = a = b = 90°$.
35. $\alpha = \gamma = b = 90°$.
36. $a = b = c = 90°$.

247. Isosceles and quadrantal triangles. Suppose that the spherical triangle ABC is isosceles, with $a = b$; then, it follows that $\alpha = \beta$. If a great circle is passed through C and the mid-point D of AB, then triangles ADC and BDC are symmetric right triangles with right angles at D. By first solving one of these right triangles, we can solve triangle ABC if any two of its distinct parts (a, α, c, γ) are given.

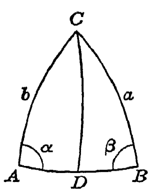


FIG. 7

Illustration 1. In Figure 7, if we are given $a = 75°$, $b = 75°$, and $\gamma = 58°$, then $\angle ACD = 29°$. Hence, we can find α and AD by solving $\triangle ADC$ by use of (I) to (X). Then $c = 2(AD)$.

NOTE 1. Let ABC be any spherical triangle and let $A'B'C'$ be its polar triangle. Then, if a', b', c', α', β', and γ' are the parts of $A'B'C'$, from Section 191 we recall that

$$\begin{array}{lll} a' = 180° - \alpha; & b' = 180° - \beta; & c' = 180° - \gamma; \\ \alpha' = 180° - a; & \beta' = 180° - b; & \gamma' = 180° - c. \end{array} \qquad (1)$$

A *quadrantal triangle* ABC is a triangle with at least one *side* equal to 90°; suppose that the triangle is lettered so that $c = 90°$. Then, from (1), in the polar triangle $A'B'C'$, $\gamma' = 180° - 90° = 90°$. Thus, $A'B'C'$ is a *right* spherical triangle. Hence, if we are given two parts of a quadrantal triangle in addition to the 90° side, we can solve the triangle by first solving the polar triangle and then using (1).

Example 1. Solve triangle ABC if $c = 90°$, $\gamma = 43°$, and $a = 132°$.

Solution. 1. From (1), in the polar triangle $A'B'C'$,

$$\gamma' = 180° - c = 90°; \quad c' = 180° - \gamma = 137°; \quad \alpha' = 180° - a = 48°.$$

2. On solving triangle $A'B'C'$ by use of Napier's rules, we obtain $a' = 30°\ 27'$; $b' = 148°\ 2'$; $\beta' = 129°\ 5'$. Hence, from (1),

$$\alpha = 180° - a' = 149°\ 33'; \quad \beta = 180° - b' = 31°\ 58'; \quad b = 50°\ 55'.$$

EXERCISE 3

*Solve the oblique spherical triangle by use of five-place logarithms.**

1. $a = b = 80°\ 32'$; $c = 142°\ 40'$.
2. $b = c = \quad 16'$; $a = 98°\ 36'$.
3. $\alpha = 94°\ 30'$; $b = c = 53°\ 27'$.
4. $\alpha = \beta = 48°\ 21'$; $\gamma = 112°\ 18'$.
5. $\beta = \gamma = 53°\ 49'$; $c = 37°\ 32'$.
6. $c = 90°$; $a = 28°\ 47'$; $b = 63°\ 57'$.
7. $b = 90°$; $\alpha = 128°\ 29'$; $\gamma = 133°\ 45'$.
8. $a = 90°$; $b = 121°\ 45'$; $\alpha = 143°\ 26'$.
9. $\alpha = \beta = 67°\ 19.8'$; $c = 94°\ 15.4'$.
10. $a = 90°$; $\alpha = 46°\ 17'\ 8''$; $b = 64°\ 11'\ 40''$.

*Solve the oblique triangle ABC by use of auxiliary right triangles and five-place logarithms.**

11. $b = 48°\ 27'$; $\alpha = 73°\ 29'$; $c = 128°\ 33'$.

HINT. Pass a great circle through C perpendicular to AB and solve the resulting right spherical triangles. First construct $\triangle ABC$ approximately to scale by starting with side c in a horizontal plane.

12. $a = 99°\ 26'$; $c = 43°\ 14'$; $\beta = 36°\ 57'$.
13. $\beta = 57°\ 18'$; $c = 67°\ 26'$; $a = 18°\ 29'$.
14. $\gamma = 23°\ 17'$; $\beta = 54°\ 16'$; $a = 115°\ 18'$.

NOTE 1. If the class is not to study oblique spherical triangles, nevertheless all applications in Chapter XVII can be taken up after Exercise 3 (except for Case I of Section 272). Under these circumstances, the oblique triangles involved would be solved as in Problem 11 of Exercise 3.

* Four-place answers are given also, except for Problems 9 and 10.

Chapter XVI

OBLIQUE SPHERICAL TRIANGLES

248. The law of sines. *In any spherical triangle ABC, the sines of the sides are proportional to the sines of the corresponding opposite angles.* That is,

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma} \tag{I}$$

Proof. 1. Draw a great circle through B perpendicular to the plane of AC. This circle cuts AC at a point D (see Figure 8), or else cuts AC *extended* at a point D where $AD < 180°$ (see Figure 9). Then, triangles ABD and CDB have right angles at D. In each figure, let the measure of BD be h.

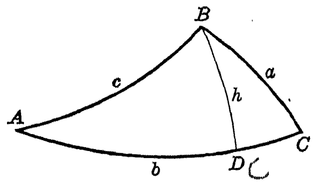


FIG. 8

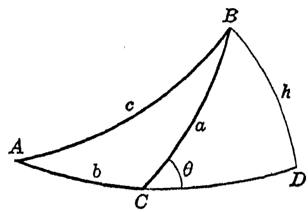


FIG. 9

2. By use of formula (I), Section 243,

for $\triangle ADB$, $\qquad \sin h = \sin \alpha \sin c;$ (1)

for $\triangle DBC$, in Figure 8, $\qquad \sin h = \sin \gamma \sin a;$ (2)

for $\triangle DBC$, in Figure 9, $\qquad \sin h = \sin \theta \sin a.$ (3)

3. In Figure 9, $\theta = 180° - \gamma$; hence, $\sin \theta = \sin \gamma$ and (3) becomes (2). Thus, (1) and (2) are true for both figures; hence,

$$\sin \alpha \sin c = \sin \gamma \sin a, \text{ or } \quad \frac{\sin a}{\sin \alpha} \quad \frac{\sin c}{\sin \gamma} \tag{4}$$

4. Similarly, by passing a great circle through C perpendicular to AB, we could show that

$$\frac{\sin a}{\sin \alpha} \quad \frac{\sin b}{\sin} \tag{5}$$

Equations (4) and (5) are equivalent to formulas (I).

249. The law of cosines. *In any spherical triangle ABC,*

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha; \tag{II}$$

$$\cos b = \cos a \cos c + \sin a \sin c \cos \beta; \tag{III}$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma. \tag{IV}$$

Proof. **(For the case of Figure 8.)** By use of (V), Section 243, applied to $\triangle\!\!\!\triangle BDC$ and ADB, since $DC = (b - AD)$ we obtain

$$\cos a = \cos h \cos (b - AD); \qquad \cos c = \cos h \cos AD. \tag{1}$$

$$\frac{\cos a}{\cos c} = \frac{\cos h \cos (b - AD)}{\cos h \cos AD} = \frac{\cos b \cos AD + \sin b \sin AD}{\cos AD}.$$

$$\frac{\cos a}{\cos c} = \cos b + \sin b \tan AD. \tag{2}$$

From (II), Section 243, for $\triangle ABD$, we obtain $\tan AD = \tan c \cos \alpha$. Therefore, from (2),

$$\cos a = \cos b \cos c + \sin b \cos c \tan c \cos \alpha;$$

or, $$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$$

NOTE 1. In Figure 9, $CD = AD - b = -(b - AD)$. On going through the details of the preceding proof and recalling the identity $\cos \theta = \cos (-\theta)$, we again obtain (II).

By interchanging the roles of the letters in (II), or by stating that result *in words* as relating to *any* side, called a for *convenience*, we obtain (III) and (IV).

We call (II), (III), and (IV) the **law of cosines for sides.** On applying these formulas to the *polar* triangle $A'B'C'$ of triangle ABC, we obtain the following formulas, called the **law of cosines for angles:**

$$\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a; \tag{V}$$

$$\cos \beta = -\cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos b; \tag{VI}$$

$$\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c. \tag{VII}$$

Proof. For triangle $A'B'C'$, from (II) we obtain

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos \alpha'. \tag{3}$$

Recall that

$$a' = 180° - \alpha; \quad b' = 180° - \beta; \quad c' = 180° - \gamma; \quad \alpha' = 180° - a.$$

Hence, $\sin \alpha' = \sin a$; $\cos a' = -\cos \alpha$; $\sin b' = \sin \beta$; etc. Therefore, from (3) we obtain (V). Similarly, we obtain (VI) and (VII).

250. Formulas for the half-sides and half-angles. Let

$$s = \tfrac{1}{2}(a + b + c) \text{ and } r = \sqrt{\frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}}; \tag{1}$$

then, we shall prove that, for any spherical triangle ABC,

$$\tan \frac{\alpha}{2} = \frac{r}{\sin (s - a)}; \quad \tan \frac{\beta}{2} = \frac{r}{\sin (s - b)}; \quad \tan \frac{\gamma}{2} = \frac{r}{\sin (s - c)}. \tag{VIII}$$

★NOTE 1. From solid geometry, we recall that the sum of the sides of a spherical triangle is less than 360°, and that the sum of any two sides is greater than the third side. Hence, $a + b + c < 360°$; thus, $s < 180°$ and $\sin s > 0°$. Also,

$$\tfrac{1}{2}(b + c - a) = s - a; \quad \tfrac{1}{2}(a + c - b) = s - b; \quad \tfrac{1}{2}(a + b - c) = s - c. \tag{2}$$

Since $s < 180°$ and $b + c > a$, hence $0 < s - a < 180°$; $\sin (s - a) > 0$; $\sin (s - b) > 0$; $\sin (s - c) > 0$. Therefore, the expression under the radical in (1) is *positive*, and r is a *real* number.

NOTE 2. Recall the following trigonometric identities:

$$\cos (\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi; \tag{3}$$

$$\cos x - \cos y = 2 \sin \tfrac{1}{2}(y + x) \sin \tfrac{1}{2}(y - x). \tag{4}$$

Proof of (VIII), for $\tfrac{1}{2}\alpha$. **1.** From (II), $\cos \alpha = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$.

Hence, $$1 - \cos \alpha = \frac{(\cos b \cos c + \sin b \sin c) - \cos a}{\sin b \sin c}$$

[From (3)] $$= \frac{\cos (b - c) - \cos a}{\sin b \sin c}$$

[From (4)] $$= \frac{2 \sin \frac{1}{2}(a + b - c) \sin \frac{1}{2}(a - b + c)}{\sin b \sin c};$$

[Using (2)] $$1 - \cos \alpha = \frac{2 \sin (s - c) \sin (s - b)}{\sin b \sin c}. \tag{5}$$

2. Similarly, we find that $$1 + \cos \alpha = \frac{2 \sin s \sin (s - a)}{\sin b \sin c}. \tag{6}$$

3. From the formula for the tangent of half of an angle α in terms of $\cos \alpha$,

$$\tan \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}} \tag{7}$$

$$\sqrt{\frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}} \qquad \frac{}{\sin^2 (s - a)} = \frac{}{\sin (s - a)}.$$

By applying (VIII) to the polar triangle $A'B'C'$, if we let

$$S = \tfrac{1}{2}(\alpha + \beta + \gamma) \text{ and } R = \sqrt{\frac{-\cos S}{\cos(S-\alpha)\cos(S-\beta)\cos(S-\gamma)}}, \quad (8)$$

we can prove that

$$\left.\begin{aligned} \tan \tfrac{1}{2}a &= R\cos(S-\alpha);\\ \tan \tfrac{1}{2}b &= R\cos(S-\beta);\\ \tan \tfrac{1}{2}c &= R\cos(S-\gamma). \end{aligned}\right\} \quad \text{(IX)}$$

Proof of (IX). **1.** For the polar triangle $A'B'C'$, $s' = \frac{1}{2}(a' + b' + c')$;

$$\sqrt{\frac{\sin(s'-a')\sin(s'-b')\sin(s'-c')}{\sin s'}}; \quad (9)$$

$$\tan\frac{\alpha'}{2} = \frac{r'}{\sin(s'-a')}. \quad (10)$$

2. On using $\alpha' = 180° - a$, $a' = 180° - \alpha$, etc., we obtain

$$s' = \tfrac{1}{2}[540° - (\alpha + \beta + \gamma)] = 270° - S;$$

$$s' - a' = \tfrac{1}{2}(b' + c' - a') = 90° - (S - \alpha); \quad (11)$$

$$s' - b' = 90° - (S - \beta); \quad s' - c' = 90° - (S - \gamma); \quad \tfrac{1}{2}\alpha' = 90° - \tfrac{1}{2}a.$$

Hence, $\tan \frac{1}{2}\alpha' = \cot \frac{1}{2}a$; $\sin s' = -\cos S$;

$$\sin(s' - a') = \cos(S - \alpha);$$

$$\sin(s' - b') = \cos(S - \beta); \quad \sin(s' - c') = \cos(S - \gamma).$$

3. On using the preceding relations in (9) and (10), we find that

$$r' = \frac{1}{R} \text{ and } \cot\frac{a}{2} = \frac{1}{R\cos(S-\alpha)}; \text{ or, } \tan\frac{a}{2} = R\cos(S-\alpha).$$

★Note 3. From Section 192, the sum of the angles of a spherical triangle lies between 180° and 540°. Thus, $180° < \alpha + \beta + \gamma < 540°$ or $90° < S < 270°$ and $\cos S$ is *negative*. Moreover, from Note 1 it is seen that $0 < (s' - a') < 180°$ and, from (11), $(S - \alpha) = 90° - (s' - a')$. Hence, $(S - \alpha)$ is *greater than* $-90°$ but *less than* $90°$ and therefore $\cos(S - \alpha)$ is *positive*. Thus, the radicand in (8) is always positive.

Since $\sin^2 \frac{1}{2}\alpha = \frac{1}{2}(1 - \cos\alpha)$ and $\cos^2 \frac{1}{2}\alpha = \frac{1}{2}(1 + \cos\alpha)$, from (5) and (6) we obtain the following formulas (X) and, by symmetry, corresponding formulas relating to β and γ.

$$\sin\frac{\alpha}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}; \quad \cos\frac{\alpha}{2} = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}. \quad \text{(X)}$$

$$\sin \frac{\beta}{2} = \sqrt{\frac{\sin (s - a) \sin (s - c)}{\sin a \sin c}}; \quad \cos \frac{\beta}{2} = \sqrt{\frac{\sin s \sin (s - b)}{\sin a \sin c}}. \quad \text{(XI)}$$

$$\sin \frac{\gamma}{2} = \sqrt{\frac{\sin (s - a) \sin (s - b)}{\sin a \sin b}}; \quad \cos \frac{\gamma}{2} = \sqrt{\frac{\sin s \sin (s - c)}{\sin a \sin b}}. \quad \text{(XII)}$$

251. Gauss's formulas, or Delambre's analogies.* If α and β are any two angles of a spherical triangle ABC, then

$$\sin \tfrac{1}{2}(\alpha + \beta) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}c} \cos \tfrac{1}{2}\gamma; \qquad \text{(XIII)}$$

$$\sin \tfrac{1}{2}(\alpha - \beta) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}c} \cos \tfrac{1}{2}\gamma; \qquad \text{(XIV)}$$

$$\cos \tfrac{1}{2}(\alpha + \beta) = \frac{\cos \frac{1}{2}(a + b)}{\cos \frac{1}{2}c} \sin \tfrac{1}{2}\gamma; \qquad \text{(XV)}$$

$$\cos \tfrac{1}{2}(\alpha - \beta) = \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}c} \sin \tfrac{1}{2}\gamma. \qquad \text{(XVI)}$$

Proof *of Delambre's analogies.* **1.** By use of an addition formula,

$$\sin \tfrac{1}{2}(\alpha + \beta) = \sin (\tfrac{1}{2}\alpha + \tfrac{1}{2}\beta) = \sin \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta + \cos \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta. \qquad (1)$$

2. From (X), (XI), and (XII),

$$\sin \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta = \frac{\sin (s - b)}{\sin c}\sqrt{\frac{\sin s \sin (s - c)}{\sin a \sin b}} = \frac{\sin (s - b)}{\sin c} \cos \tfrac{1}{2}\gamma.$$

Similarly, $$\cos \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta = \frac{\sin (s - a)}{\sin c} \cos \tfrac{1}{2}\gamma.$$

3. From (1) and Step 2,

$$\sin \tfrac{1}{2}(\alpha + \beta) = \frac{\sin (s - b) + \sin (s - a)}{\sin c} \cos \tfrac{1}{2}\gamma$$

$$= \frac{2 \sin \frac{1}{2}(2s - a - b) \cos \frac{1}{2}(a - b)}{2 \sin \frac{1}{2}c \cos \frac{1}{2}c} \cos \tfrac{1}{2}\gamma. \qquad (2)$$

In (2), we used the identity

$$\sin x + \sin y = 2 \sin \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y).$$

4. Since $2s - a - b = a + b + c - a - b = c$, (2) gives (XIII).

5. Similarly, we are led to (XIV), (XV), and (XVI). By symmetrical changes of letters in (XIII) to (XVI) we may write the Delambre analogies involving α and γ or β and γ in the left members.

* The word *analogy* here is used in an old sense meaning a *proportion.*

NOTE 1. Each of Delambre's analogies involves all six parts of triangle ABC. Hence, these formulas are useful in checking the solutions of triangles.

252. Napier's analogies. The following four formulas and the eight symmetrical equations involving $(\beta, \gamma;\ b, c)$ or $(\alpha, \gamma;\ a, c)$ in the left members are called **Napier's analogies.** Each of these formulas involves five parts of triangle ABC. We obtain (XVII) by dividing each side of (XIV) by the corresponding side of (XIII); etc.

$$\text{(XVII)}\quad \frac{\sin \frac{1}{2}(\alpha-\beta)}{\sin \frac{1}{2}(\alpha+\beta)}=\frac{\tan \frac{1}{2}(a-b)}{\tan \frac{1}{2}c}. \qquad \text{(XIX)}\quad \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)}=\frac{\tan \frac{1}{2}(\alpha-\beta)}{\cot \frac{1}{2}\gamma}.$$

$$\text{(XVIII)}\quad \frac{\cos \frac{1}{2}(\alpha-\beta)}{\cos \frac{1}{2}(\alpha+\beta)}=\frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}c}. \qquad \text{(XX)}\quad \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)}=\frac{\tan \frac{1}{2}(\alpha+\beta)}{\cot \frac{1}{2}\gamma}.$$

★EXERCISE 4

1. By use of figures like Figures 8 and 9, prove that the sines of b and c are proportional to the sines of β and γ.
2. By use of the figures used in Problem 1, prove (III) or (IV).
3. Prove (VI) by use of (III). 4. Prove (VII) by use of (IV).
5. Completely verify equation (6) in Section 250.
6. Derive the formula for $\tan \frac{1}{2}\beta$ in (VIII); from this result prove the formula for $\tan \frac{1}{2}b$ in (IX).
7. Prove (XIV) and (XVI). 8. Prove (XV) for $\cos \frac{1}{2}(\alpha + \gamma)$.

253. Solution of oblique triangles. The following cases arise.

I. *Given three sides.*
II. *Given three angles.*
III. *Given two sides and the included angle.*
IV. *Given two angles and the included side.*
V. *Given two sides and an angle opposite one of these sides.*
VI. *Given two angles and a side opposite one of these angles.*

We shall find that formulas (I), (VIII), and (IX), and Napier's analogies are sufficient for the solution of all cases.

Case I; given three sides. The angles are found by use of formulas (VIII). The solution may be checked by the law of sines, or by one of Napier's analogies involving all angles.

Case II; given three angles. The sides are found by use of formulas (IX). The solution may be checked by the law of sines or by one of Napier's analogies involving all the sides.

Example 1. Solve the triangle ABC if α $123° 56'$, $\beta = 72° 20'$, and $\gamma = 81° 40'$.

Formulas	Computation
$S = \frac{1}{2}(\alpha + \beta + \gamma)$. To check here, notice $(S - \alpha) + (S - \beta) + (S - \gamma)$ $= 3S - (\alpha + \beta + \gamma) = S$.	$\left.\begin{aligned}\alpha &= 123° 56'\\ \beta &= 72° 20'\\ \gamma &= 81° 40'\end{aligned}\right\}(+)$ $\left.\begin{aligned}S-\alpha &= 15° \ 2'\\ S-\beta &= 66° 38'\\ S-\gamma &= 57° 18'\end{aligned}\right\}(+)$ $2S = 277° 56'$ $S = 138° 58'$
$R = \sqrt{\dfrac{-\cos S}{\cos(S-\alpha)\cos(S-\beta)\cos(S-\gamma)}}$. $\cos 138° 58' = -\cos 41° 2'$; $-\cos S = \cos 41° 2'$.	$\log\cos(S-\alpha) = 9.9848 - 10$ $\log\cos(S-\beta) = 9.5984 - 10$ (+) $\log\cos(S-\gamma) = 9.7326 - 10$ $\log$ denom. $= 9.3158 - 10$ $\log(-\cos S) = 9.8776 - 10$ (−) $\log R^2 = 0.5618$ (÷ by 2)
$\tan\frac{1}{2}a = R\cos(S - \alpha)$.	$\log R = 0.2809$ $\log\cos(S-\alpha) = 9.9848 - 10$ (+) $\log\tan\frac{1}{2}a = 0.2657$; $\frac{1}{2}a = 61° 32'$; $a = 123° 4'$.
$\tan\frac{1}{2}b = R\cos(S - \beta)$. $\tan\frac{1}{2}c = R\cos(S - \gamma)$.	Similarly, we find $b = 74° 16'$; $c = 91° 46'$.
Summary.	$123° 4'$; $b = 74° 16'$; $c = 91° 46'$.

Check. $\dfrac{\sin\alpha}{\sin a} = \dfrac{\sin\beta}{\sin b} = \dfrac{\sin\gamma}{\sin c}$.

$\log\sin\alpha = 9.9189 - 10$	$\log\sin\beta = 9.9790 - 10$	$\log\sin\gamma = 9.9954 - 10$
$\log\sin a = 9.9233 - 10(-)$	$\log\sin b = 9.9834 - 10(-)$	$\log\sin c = 9.9998 - 10(-)$
$\log$ quot. $= 9.9956 - 10$.	$\log$ quot. $= 9.9956 - 10$.	$\log$ quot. $= 9.9956 - 10$.

Comment. In Example 1, since c is near to 90°, the check by the law of sines is not useful for detecting any small error in c. A more refined test would be obtained by checking with (XVII).

NOTE 1. Case II may be solved as follows: (*i*) find the sides of the polar triangle; (*ii*) solve for its angles, by Case I; (*iii*) compute their supplements to find the sides of the original triangle.

NOTE 2. In Case I we may solve as follows: (*i*) find one angle by use of the law of cosines; (*ii*) find the other angles by use of the law of sines. A similar method is available for Case II. This method is somewhat inconvenient from the standpoint of logarithmic computation.

★NOTE 3. The method of Note 2 for Case I assumes added convenience if we first transform (II), (III), and (IV) by introducing a new trigonometric function called the **haversine,** abbreviated **hav,** and defined by the equation

$$\textbf{hav}\,\theta = \tfrac{1}{2}(1 - \cos\theta) \quad \textit{or} \quad \cos\theta = 1 - 2\,\textbf{hav}\,\theta. \tag{1}$$

In (II), let $\cos a$ and $\cos \alpha$ be expressed in terms of hav a and hav α. Then, after using (3) of Section 250, in place of (II) we obtain

$$\textbf{hav}\, a = \textbf{hav}\, (b - c) + \textbf{sin}\, b\, \textbf{sin}\, c\, \textbf{hav}\, \alpha. \qquad \text{(II)}'$$

Certain other formulas of this chapter can also be altered conveniently by use of haversines. Such formulas are frequently useful in navigation, if a table of values of the haversine is available.

EXERCISE 5

Solve and check by use of four-place or five-place logarithms.

1. $a=39°$; $b=49°$; $c=62°$.
2. $a=61°$; $b=43°$; $c=68°$.
3. $\alpha=125°$; $\beta=73°$; $\gamma=84°$.
4. $\alpha=75°$; $\beta=83°$; $\gamma=69°$.
5. $a=81° 27'$; $b=38° 50'$; $c=92° 43'$.
6. $\alpha=116° 19'$; $\beta=55° 30'$; $\gamma=80° 37'$.
7. $\alpha=124° 16'$; $\beta=53° 40'$; $\gamma=51° 24'$.
8. $a=147° 40'$; $b=72° 10'$; $c=121° 36'$.

Solve and check by use of five-place logarithms.

9. $a = 48° 57.6'$; $b = 69° 28.4'$; $c = 83° 13.2'$.
10. $\alpha = 127° 39.4'$; $\beta = 91° 16.2'$; $\gamma = 75° 36.6'$.
11. $\alpha = 138° 26.7'$; $\beta = 77° 27.3'$; $\gamma = 81° 13.0'$.
12. $a = 48° 42.8'$; $b = 106° 54.2'$; $c = 84° 19.0'$.

★**254. Rules for species.** In the use of the law of sines to determine an unknown side or angle, usually two values are obtained for the unknown part. To group ambiguous results properly, or to reject impossible values, *we usually employ Theorem* II *in Section* 245. Sometimes, however, the following theorems are useful.

Theorem I. *Half the sum of any two sides is of the same species as half the sum of the opposite angles.*

Theorem II. *A first side (or angle) which differs from* 90° *more than another side (or angle) has the same species as the angle (or side) opposite the first side (or angle).*

NOTE 1. The student should prove Theorem I by use of (XVIII). If formula (II) is solved for $\cos \alpha$, proper consideration of the result would establish Theorem II.

255. Case III; given two sides and the included angle. If the given parts are (a, b, γ), we find α and β by use of (XIX) and (XX), and c by use of (XVII). The solution may be checked by the law of sines, or by one of Gauss's formulas.

Example 1. Solve the triangle ABC if $a = 125° 38'$, $c = 73° 24'$, and $\beta = 102° 16'$.

Formulas and Data	Computation
Data: $\left.\begin{array}{l} a = 125^\circ 38'; \\ c = \;\; 73^\circ 24'. \end{array}\right\} \beta = 102^\circ 16'.$	$a - c = 52^\circ 14'; \quad a + c = 199^\circ\ 2';$ $\frac{1}{2}(a - c) = 26^\circ\ 7'; \quad \frac{1}{2}(a + c) = 99^\circ 31';$ $\frac{1}{2}\beta = 51^\circ 8'.$
From (XIX): $\tan \frac{1}{2}(\alpha - \gamma) = \dfrac{\cot \frac{1}{2}\beta \sin \frac{1}{2}(a - c)}{\sin \frac{1}{2}(a + c)}.$ $\sin 99^\circ 31' = \sin 80^\circ 29'.$	$\log \cot \frac{1}{2}\beta = 9.9063 - 10$ $\log \sin \frac{1}{2}(a - c) = 9.6436 - 10 \quad (+)$ $\log \text{product} = 19.5499 - 20$ $\log \sin \frac{1}{2}(a + c) = 9.9940 - 10 \quad (-)$ $\log \tan \frac{1}{2}(\alpha - \gamma) = 9.5559 - 10;$ $\frac{1}{2}(\alpha - \gamma) = 19^\circ 47'.$
From (XX): $\tan \frac{1}{2}(\alpha + \gamma) = \dfrac{\cot \frac{1}{2}\beta \cos \frac{1}{2}(a - c)}{\cos \frac{1}{2}(a + c)}.$ $\cos 99^\circ 31' = -\cos 80^\circ 29'.$	$\log \cot \frac{1}{2}\beta = 9.9063 - 10$ $\log \cos \frac{1}{2}(a - c) = 9.9532 - 10 \quad (+)$ $\log \text{product} = 9.8595 - 10$ $(n) \log \cos \frac{1}{2}(a + c) = 9.2184 - 10 \quad (-)$ $(n) \log \tan \frac{1}{2}(\alpha + \gamma) = 0.6411;$ from Table VI, $0.6411 = \log \tan 77^\circ 8'.$ Hence, $\frac{1}{2}(\alpha + \gamma) = 102^\circ 52'.$ $\frac{1}{2}(\alpha - \gamma) = 19^\circ 47'.$ (From above) $\alpha = 122^\circ 39'.$ [Adding] $\gamma = 83^\circ\ 5'.$ [Subtracting]
From (XVII): $\tan \frac{1}{2}b = \dfrac{\tan \frac{1}{2}(a - c) \sin \frac{1}{2}(\alpha + \gamma)}{\sin \frac{1}{2}(\alpha - \gamma)}.$	The student may verify that $\frac{1}{2}b = 54^\circ 41'; \quad b = 109^\circ 22'.$

Check. We use formula XV:

$$\frac{\cos \frac{1}{2}(\alpha + \beta)}{\sin \frac{1}{2}\gamma} \qquad \frac{\cos \frac{1}{2}(a + b)}{\cos \frac{1}{2}c}.$$

$$\begin{array}{rl} (n) \log \cos \frac{1}{2}(\alpha + \beta) &= 9.5822 - 10 \\ (-) \log \sin \frac{1}{2}\gamma &= 9.8216 - 10 \\ \hline (n) \log \text{quotient} &= 9.7606 - 10. \end{array} \qquad \begin{array}{rl} (n) \log \cos \frac{1}{2}(a + b) &= 9.6644 - 10 \\ (-) \log \cos \frac{1}{2}c &= 9.9040 - 10 \\ \hline (n) \log \text{quotient} &= 9.7604 - 10. \end{array}$$

NOTE 1. In an example under Case III, the unknown side could be found by using the law of sines instead of a Napier analogy. However, the Napier analogy gives the side *without any ambiguity* while the law of sines would give *two* values for the side.

NOTE 2. In Case III, we may also solve as follows: (*i*) use the law of cosines to find the third side; (*ii*) use the law of sines to find the unknown angles.*

256. Case IV; given two angles and the included side. If the given parts are α, β, and c, we use (XVII) and (XVIII) to find a and b and (XIX) to find γ. The solution may be checked by the law of sines, or by one of Gauss's formulas. The details are similar to those met in considering Case III.

* Or, if a table of haversines is available, use (II)′ of Section 253.

NOTE 1. In Case IV, we can also solve by use of the polar triangle, for which the data would come under Case III.

EXERCISE 6

Solve the spherical triangle ABC by use of four-place or five-place logarithms and check the solution.

1. $a = 93° 8'$; $b = 46° 4'$; $\gamma = 71° 6'$.
2. $a = 58° 6'$; $b = 22° 8'$; $\gamma = 112° 0'$.
3. $\alpha = 118° 4'$; $\beta = 36° 2'$; $c = 35° 6'$.
4. $\alpha = 108° 46'$ $\beta = 40° 48'$; $c = 80° 14'$.
5. $b = 127° 46'$ $c = 65° 32'$; $\alpha = 94° 38'$.
6. $\alpha = 126° 34'$ $\gamma = 106° 24'$; $b = 121° 47'$.
7. $\beta = 44° 28'$; $\gamma = 116° 24'$; $a = 63° 29'$.
8. $a = 64° 27'$; $c = 104° 39'$; $\beta = 64° 20'$.

Solve by use of five-place logarithms and check the solution.

9. $a = 124° 17.8'$ $c = 83° 16.6'$; $\beta = 63° 19.6'$.
10. $\alpha = 124° 16.3'$ $\gamma = 52° 19.7'$; $b = 56° 38.3'$.
11. $\beta = 101° 16.7'$ $\gamma = 143° 25.2'$; $a = 104° 16.0'$.
12. $b = 84° 28.2'$; $c = 137° 41' 44''$; $\alpha = 112° 10' 33''$.

257. Case V, the ambiguous case; given two sides and an angle opposite one of these sides. If the given parts are a, b, and α, the geometric possibilities bear a close resemblance to those for plane triangles in the ambiguous case. For the illustration in Figure 10, there are two solutions, AB_1C and AB_2C. For any data, there will be either *no* solution, or *just one*, or *just two* solutions, depending on the values of the given parts. If a, b, and α are given, we obtain β from the law of sines,

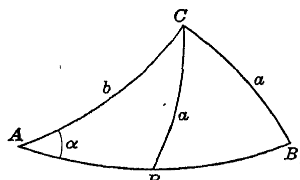


FIG. 10

$$\sin \beta = \frac{\sin b \sin \alpha}{\sin a} \tag{1}$$

Corresponding to each permissible value of β obtained from (1), we can find *just one* corresponding pair of values for the unknown side c and its opposite angle γ by use of the Napier analogies which involve $\frac{1}{2}(a \pm b)$ and $\frac{1}{2}(\alpha \pm \beta)$.

In considering equation (1) we meet the following cases.

i. If $\sin \beta > 1$, *there is* ***no*** *solution.*

ii. If $\sin \beta = 1$, *there is* ***just one*** *solution, with* $\beta = 90°$.

iii. If $\sin \beta < 1$, *we find* **an acute value** β_1 *from our tables and* **a second value** $\beta_2 = 180° - \beta_1$. *Each of* β_1 *and* β_2 *must be tested by Theorem* II *on page* 219 *to reject inadmissible values. Either* **both,** *or* **just one,** *or* **neither** *of* β_1 *and* β_2 *may be admissible.*

Example 1. Solve triangle ABC if $a = 46° 39'$; $b = 33° 7'$; $\beta = 44° 28'$.

Solution. The figure for the problem would look like Figure 10, with the letters A and B interchanged and the other corresponding alterations.

To find α and the number of solutions:

$\sin \alpha = \dfrac{\sin \beta \sin a}{\sin b}$.	$\log \sin \beta = 9.8454 - 10$
	$\log \sin a = 9.8617 - 10 \quad (+)$
	$\log \text{product} = 19.7071 - 20$
	$\log \sin b = 9.7374 - 10 \quad (-)$
	$\log \sin \alpha = 9.9697 - 10$; hence,
	$\alpha = \mathbf{68° 50'}$ or $\alpha = \mathbf{111° 10'}$

By Theorem II, Section 245, since $b < a$, hence we must have $\beta < \alpha$. We verify that $\beta < 68° 50'$ and $\beta < 111° 10'$. Therefore, both α_1 and α_2 are permissible; there are *two* solutions.

To find c and γ we use (XVIII) and (XIX):

$$\tan \tfrac{1}{2}c = \frac{\tan \frac{1}{2}(a + b) \cos \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)}; \qquad \cot \tfrac{1}{2}\gamma = \frac{\tan \frac{1}{2}(\alpha - \beta) \sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}(a - b)}.$$

$a = 46° 39'$	$\alpha_1 = 68° 50'$	$\alpha_2 = 111° 10'$
$b = 33° 7'$	$\beta = 44° 28'$	$\beta = 44° 28'$
$a + b = 79° 46'$	$\alpha_1 + \beta = 113° 18'$	$\alpha_2 + \beta = 155° 38'$
$a - b = 13° 32'$	$\alpha_1 - \beta = 24° 22'$	$\alpha_2 - \beta = 66° 42'$
$\frac{1}{2}(a + b) = 39° 53'$	$\frac{1}{2}(\alpha_1 + \beta) = 56° 39'$	$\frac{1}{2}(\alpha_2 + \beta) = 77° 49'$
$\frac{1}{2}(a - b) = 6° 46'$	$\frac{1}{2}(\alpha_1 - \beta) = 12° 11'$	$\frac{1}{2}(\alpha_2 - \beta) = 33° 21'$
To compute c:	**By use of α_1:**	**By use of α_2:**
$\log \tan \frac{1}{2}(a + b) =$	$9.9220 - 10$	$9.9220 - 10$
$\log \cos \frac{1}{2}(\alpha + \beta) =$	$9.7402 - 10 \quad (+)$	$9.3244 - 10 \quad (-)$
$\log \text{product} =$	$9.6622 - 10$	$9.2464 - 10$
$\log \cos \frac{1}{2}(\alpha - \beta) =$	$9.9901 - 10 \quad (-)$	$9.9218 - 10 \quad (-)$
$\log \tan \frac{1}{2}c =$	$9.6721 - 10$	$9.3246 - 10$
Hence,	$\frac{1}{2}c_1 = 25° 10'$; $c_1 = 50° 20'$.	$\frac{1}{2}c_2 = 11° 55'$; $c_2 = 23° 50'$.

From the formula for $\cot \frac{1}{2}\gamma$ we obtain γ_1 ' $50'$; $\gamma_2 = 31° 14'$.

Summary. *First solution:* $\alpha_1 = 68° 50'$; $c_1 = 50° 20'$; $\gamma_1 = 80° 50'$.
Second solution: $\alpha_2 = 111° 10'$; $c_2 = 23° 50'$; $\gamma_2 = 31° 14'$.

Example 2. Solve the triangle ABC if $c = 31° 50'$, $b = 83° 20'$, and $\gamma = 105° 40'$.

Solution. **To find β:** $\qquad \sin \beta = \dfrac{\sin \gamma \sin b}{\sin c}$.

$$\begin{array}{rll}
\log \sin \gamma & = 9.9836 - 10 & \\
\log \sin b & = 9.9971 - 10 & (+) \\
\hline
\log \text{product} & = 9.9807 - 10 & \\
\log \sin c & = 9.7222 - 10 & (-) \\
\hline
\end{array}$$

$(\sin 105° 40' = \sin 74° 20')$

$\log \sin \beta = 0.2585$; this implies that $\sin \beta > 1$; there is no solution.

258. Case VI; given two angles and a side opposite one of these angles. The details of a solution are similar to those under Case V. There may be *no* solution, *just one*, or *just two* solutions.

Example 1. Determine how many solutions would exist for triangle ABC if $\alpha = 36° 20'$, $\gamma = 104° 20'$, and $c = 133° 10'$.

Solution. 1. **To find a:** $\qquad \sin a = \dfrac{\sin c \sin \alpha}{\sin \gamma}$.

$$\begin{array}{rll}
\log \sin c & 9.8629 - 10 & \\
\log \sin \alpha & 9.7727 - 10 & (+) \\
\hline
\log \text{product} & = 9.6356 - 10 & \\
\log \sin \gamma & = 9.9863 - 10 & (-) \\
\hline
\end{array}$$

$(\sin 133° 10' = \sin 46° 50')$

$(\sin 104° 20' \quad \sin 75° 40')$

$\log \sin a = 9.6493 - 10$; hence $a_1 = 26° 29'$ and $a_2 = 153° 31'$.

2. **Test of the values of a by use of Theorem II, Section 245.** Since $\gamma > \alpha$, hence $c > a$. We verify that $c > 26° 29'$ but $c < 153° 31'$. Therefore, a_2 is *not admissible*. There is just one solution for triangle ABC, with $a = 26° 29'$.

EXERCISE 7

Only find the number of solutions for the spherical triangle ABC.

1. $a = 60°$; $b = 30°$; $\beta = 30°$.
2. $b = 75°$; $c = 75°$; $\gamma = 40°$.
3. $a = 30°$; $b = 60°$; $\alpha = 150°$.
4. $a = 120°$; $\alpha = 60°$; $\beta = 30°$.
5. $c = 45°$; $\beta = 60°$; $\gamma = 30°$.
6. $b = 150°$; $\beta = 45°$; $\gamma = 120°$.
7. $a = 120°$; $c = 150°$; $\gamma = 75°$.
8. $a = 30°$; $\alpha = 75°$; $\gamma = 15°$.

Solve by use of four-place or five-place logarithms.

9. $a = 47° 20'$; $b = 34° 9'$; $\beta = 43° 20'$.
10. $b = 49° 20'$; $c = 37° 26'$; $\gamma = 42° 50'$.
11. $b = 39° 14'$; $c = 67° 20'$; $\gamma = 115° 10'$.
12. $a = 53° 30'$; $b = 82° 10'$; $\beta = 118° 16'$.
13. $c = 44° 50'$; $\beta = 61° 27'$; $\gamma = 18° 10'$.
14. $a = 39° 40'$; $\alpha = 105° 10'$; $\gamma = 143° 14'$.
15. $c = 147° 20'$; $\beta = 68° 12'$; $\gamma = 27° 10'$.

16. $a = 129° 18'$; $b = 26° 40'$; $\beta = 59° 30'$.

17. $a = 132° 40'$; $b = 151° 10'$; $\alpha = 53° 15'$.

18. $c = 138° 23'$; $\alpha = 121° 30'$; $\gamma = 133° 10'$.

Solve by use of five-place logarithms.

19. $b = 103° 14.8'$; $c = 62° 31.4'$; $\gamma = 63° 20.7'$.

20. $a = 119° 31.6'$; $\alpha = 77° 43.3'$; $\beta = 52° 24.3'$.

21. $b = 57° 23.2'$; $\beta = 68° 19.4'$; $\gamma = 115° 16.8'$.

★259. Area of a spherical triangle. Let E be the *spherical excess* and H be the *area* of a spherical triangle ABC, on a sphere whose radius is ρ; then,

$$E = \alpha + \beta + \gamma - 180°; \textit{ the area of the sphere is } 4\pi\rho^2. \tag{1}$$

In geometry it is proved that H is equal to *the area of a lune whose angle is* $\frac{1}{2}E$. Since the *area of a lune* is to *the area of the sphere* as *the angle of the lune* is to 360°, hence, if E is measured in degrees,

$$\frac{H}{4\pi\rho^2} = \frac{\frac{1}{2}E}{360°}, \quad \text{or} \quad H = \frac{\pi\rho^2 E}{180°}. \tag{2}$$

NOTE 1. It can be proved that

$$\tan \tfrac{1}{4}E = \sqrt{\tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s - a) \tan \tfrac{1}{2}(s - b) \tan \tfrac{1}{2}(s - c)}, \tag{3}$$

which is called **L'Huilier's Theorem.** We omit the proof of (3).

★EXERCISE 8

Problems 1 to 4. Find the areas of the triangles of Problems 3, 4, 6, and 7, respectively, in Exercise 5, if $\rho = 100$ feet.

MISCELLANEOUS EXERCISE 9

Solve the oblique triangle ABC by use of four-place or five-place logarithms.

1. $a = 131° 16'$; $c = 78° 24'$; $\beta = 107° 19'$.

2. $a = 81° 19'$; $b = 103° 16'$; $c = 39° 26'$.

3. $\alpha = 126° 39'$; $\beta = 42° 18'$; $\gamma = 53° 26'$.

4. $a = 113° 15'$; $c = 71° 16'$; $\gamma = 61° 47'$.

5. $a = 116° 53'$; $\alpha = 61° 29'$; $\beta = 81° 14'$.

6. $a = 116° 21'$; $b = 77° 38'$; $c = 108° 29'$.

7. $b = 47° 26'$; $\beta = 118° 21'$; $\gamma = 142° 16'$.

8. $b = 67° 13'$; $\alpha = 47° 28'$; $\gamma = 118° 19'$.

9. $a = 143° 20'$; $\alpha = 137° 16'$; $\beta = 103° 27'$.

10. $a = 129° 45'$; $c = 61° 38'$; $\beta = 97° 26'$.

Chapter XVII

APPLICATIONS OF SPHERICAL TRIANGLES AND RELATED TOPICS IN NAVIGATION

260. Latitude and longitude. For our purposes, no appreciable error is made in assuming that the earth's surface is a sphere, whose radius is approximately 3959 miles. The earth revolves on a diameter whose intersections with the surface are called the north pole, N, and the south pole, S. If an observer stands at A on the earth, the direction of N is called *north* and of S is called *south;* if he faces *north,* the direction perpendicular to north and to the *right* is called *east* and the opposite direction is called *west.* The equator is the great circle on the earth whose poles are N and S. The **meridian** of a point A on the earth * is the semicircle NAS of the great circle through N, A, and S, as in Figure 11. The **latitude** of A is the angular measure of the arc of the meridian of A extending from the equator to A. If O is the earth's center and λ is the latitude of A, then, in Figure 11, $\lambda = \angle HOA$. Latitude is measured from 0° to 90°, north or south of the equator. The **colatitude** of A is the complement of its latitude and is the angular measure of arc AN, in Figure 11.

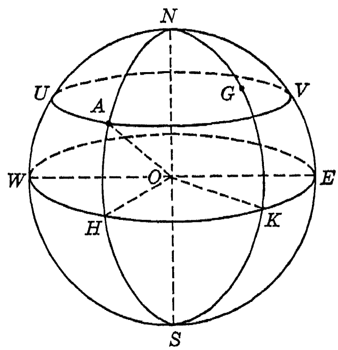


FIG. 11

Illustration 1. In Figure 11, $\lambda = 40° N$ and the colatitude of A is 50°, approximately.

All points on the earth with the same latitude λ lie on a small circle, with N and S as poles, in a plane parallel to the equator. This small circle, or **parallel of latitude,** may be called the λ-*latitude circle.* In Figure 11, we observe that the curve AUV is the parallel of latitude through A.

* *On the earth* will mean on the earth's *surface.*

The **longitude** of A is the angle of intersection of the meridian of A and the meridian of Greenwich, England, indicated by G in Figure 11. Longitude is measured from 0° to 180° east or west from Greenwich. If ϕ represents the longitude of A, then ϕ is the measure of the smallest angle formed by the planes ONA and ONG or, in Figure 11, $\phi = \angle KOH$, in the plane of the equator.

Illustration 2. In Figure 11, the longitude of A is approximately 100° W, meaning 100° west of Greenwich.

We observe that any point A on the earth is definitely located by specifying two *angular coordinates*, for A, the *latitude* and the *longitude* of A.

261. Nautical miles. A **nautical mile** is defined as the length of the arc which is subtended on a great circle on the earth by an angle of 1′ at the center of the circle:

$$(1' \textit{ of arc on a great circle on the earth}) = 1 \textit{ nautical mile;} \tag{1}$$

$$\textbf{1 nautical mile} = \textbf{6080 feet} = \textbf{1.1515 statute miles.} \tag{2}$$

To prove (2), we compute the circumference, in statute miles, of a great circle on the earth and divide this by 360° expressed in minutes:

$$360° = (360)(60') = 21{,}600 \textit{ minutes.}$$

$$(\textit{circumference of a great circle}) = 2\pi(3959) \textit{ statute miles.}$$

$$1 \textit{ nautical mile} = \frac{2\pi(3959)}{21600} \textit{ statute miles} = 1.152 \textit{ statute miles.}$$

262. Directions on the earth. The **horizontal plane** at A on the earth is an ideal plane which is *tangent to the earth at A*. In referring to a path AB, *from A to B* on the earth, we shall mean a *directed* path, from the *first* point A to the *second* point B. Any path AB has a directed tangent line which we shall call the *direction line* for AB, radiating from A in the horizontal plane. In referring to the *direction* of AB at A we mean the direction of the *direction line for AB*. In the horizontal plane, north and south are determined by the direction lines for arcs AN and AS of Figure 11. In Figure 12, AN and AS indicate these directions and we call SAN the **meridian** of A in its *horizontal plane*. In Figure 12, AB is the direction line for a

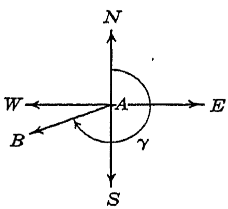


Fig. 12

path AB. In referring to the **bearing** of a point B from A on the earth, we shall mean the direction at A of the shortest great circle path AB from A to B.

In the horizontal plane, we shall describe any direction, such as AB in Figure 12, by specifying its **azimuth,** which is the angle between the north direction AN and AB, measured to the east (clockwise) from the north. The direction of the track of a ship or airplane at any point A on the track is called the **course** of the track at A. Thus, the course or any other bearing is an angle between 0° and 360°.

Illustration 1. If a ship sails on course 250°, as in Figure 12, the track AB is in a southwesterly direction. The direction of AB is 70° west of south because $(250° - 180°) = 70° = \angle BAS$.

263. Plane sailing.* Let us assume that, in the neighborhood to be considered, the earth's surface *is* the horizontal plane for some specified point A. The navigation of a ship or airplane under this approximation, which we adopt for the present, is called **plane sailing.** Instead of the actual longitude circle through A on the earth, we have the meridian NS, of Figure 12, through A on the horizontal plane. Instead of parallels of latitude on the earth, we would have corresponding straight lines perpendicular to the meridian in the horizontal plane. In Figure 12, WE represents the parallel of latitude through A.

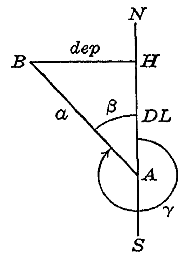


FIG. 13

In indicating the position of a point B with respect to A, we refer to the *distance* AB, designated by d, the *difference in latitude* (***DL***), and the *difference in longitude* (***DLo***), of A and B. The *departure* (***dep***) of B with respect to A is defined as the distance of B *east* or *west* of the meridian through A. In Figure 13, γ represents the course of AB and β is the acute angle between AB and the meridian.

When convenient, we shall attach signs to the departure and difference of latitude. In such a case, the departure from A to B will be considered *positive* if B is *east* and *negative* if B is *west* of A; the difference in latitude will be considered *positive* if B is *north* and *negative* if B is *south* of A. Unless otherwise stated, *dep* and *DL* may be understood to represent the numerical values of the quantities involved.

* Sections 263–268 may be omitted if the instructor wishes to proceed immediately to the applications of spherical trigonometry.

Let all distances be given in *nautical miles* unless otherwise specified. Then, by the definition of a nautical mile, if A and H are points on the earth on the same longitude circle, **the length of the great circle arc AH equals the angular measure of AH in minutes.** Hence, if we assume that AN in Figure 13 is an arc of the longitude circle through A and if DL represents the number of minutes in the angular measure of AH, then DL also is the *length* of AH in nautical miles. We shall use DL with either of these meanings.

Any problem concerning the values of d, dep, and DL is solved by plane trigonometry applied to the right triangle ABH, as illustrated in Figure 13.

$$\boldsymbol{dep = d \sin \beta; \qquad DL = d \cos \beta.} \tag{1}$$

Example 1. A ship sails for 3 hours from A at a speed of 35 nautical miles per hour (or 35 knots) on course 307° 25′. (*i*) Find the departure and difference in latitude for the final position B. (*ii*) If the latitude of A is 27° 38′ N, find the latitude of B.

Solution. In Figure 13, $\gamma = 307° 25'$, $d = 105$, and $\beta = 52° 35'$.

$$dep = 105 \sin 52° 35' = 83.4 \text{ mi.}$$

$$DL = 105 \cos 52° 35' = 63.8 \text{ mi.} = 63.8' \textit{ in angular measure;}$$

$$(\textit{latitude of } B) = (\textit{latitude of } A) + DL$$
$$= 27° 38' + 63.8' = 28° 41.8' \ (\textit{north})$$

EXERCISE 10

By plane sailing, find the departure, the difference in latitude, and the latitude of the terminal point B if a ship sails for the specified distance d on course γ from a point A with the given latitude.

1. Latitude of A is 30° 38′ N; $\gamma = 47° 20'$; $d = 240$ miles.

2. Latitude of A is 53° 16′ N; $\gamma = 143° 50'$; $d = 158$ miles.

3. Latitude of A is 15° 38′ S; $\gamma = 268° 15'$; $d = 310$ miles.

4. Latitude of A is 38° 47′ S; $\gamma = 318° 42'$; $d = 264$ miles.

By plane sailing, find the course, distance traveled from A to B, and latitude of B, for the given DL and departure, where their signs have the significance described in Section 263.

5. Latitude of A is 43° 51′ N; $DL = +152$ miles; dep $= +268$ miles.

6. Latitude of A is 28° 36′ N; $DL = -210$ miles; dep $= +165$ miles.

7. Latitude of A is 16° 43′ N; $DL = +350$ miles; dep $= -240$ miles.

8. Latitude of A is 28° 54′ S; $DL = +215$ miles; dep $= +185$ miles.

9. Latitude of A is 17° 39′ S; $DL = -125$ miles; dep $= -254$ miles.

264. Difference in longitude on a parallel of latitude. In Figure 14, A and B are two points of equal latitude, λ, on the northern hemisphere; G represents Greenwich, O the earth's center, and $WVUK$ the plane of the equator. Plane $ABHD$ is perpendicular to ON and intersects the earth's surface along the parallel of latitude through A and B. Meridians are shown through A, B, and G. The longitudes of A and B are $\angle KOV$ and $\angle KOU$, respectively. We notice that $\angle BHA = \angle UOV$ because each of these angles measures the angle between the planes $UONB$ and $VONA$. Hence, the difference of the longitudes of A and B is measured by $\angle VOU$ or, equally well, by $\angle AHB$. Therefore, *when A and B are on the same parallel of latitude, the angular measure of the shortest arc AB of this small circle is the difference of the longitudes of A and B.*

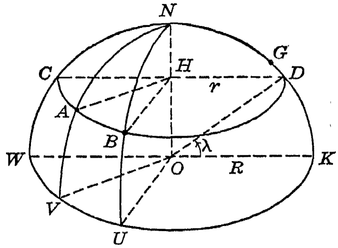


FIG. 14

265. Linear equivalent of 1′ on a parallel of latitude. In Figure 14, where λ is the latitude of D, let r be the radius of the λ-latitude circle through D and let R be the radius of the earth. Then, we have $OK = R = OD$; $r = HD$; $\angle ODH = \lambda$. Hence, from right triangle ODH, we obtain $HD = OD \cos \angle ODH$, or

$$r = R \cos \lambda. \tag{1}$$

Illustration 1. The radius of the 60°-latitude circle is

$$r = R \cos 60° = 3959 \cos 60° = 1980 \text{ statute miles.}$$

On the λ-latitude circle through D, in Figure 14, let t be *the length of an arc subtended by an angle of* 1′ *at the center H.* In a *great* circle on the earth, a central angle of 1′ subtends an arc whose length is 1 *nautical mile.* Hence, since the ratio of arcs subtended by these equal central angles equals the ratio of the radii of the circles,

$$\frac{t}{1} = \frac{r}{R}. \tag{2}$$

From (1) and (2) we obtain $t = \dfrac{R \cos \lambda}{R}$, or

$$\textbf{(1′ of arc on } \boldsymbol{\lambda}\textbf{-latitude circle)} = \boldsymbol{\cos \lambda} \textbf{ nautical miles.} \tag{3}$$

Illustration 2. From (3), if $\lambda = 60°$, (1′ of arc) $= \cos 60°$ *miles* or $\frac{1}{2}$ *mile.*

266. Parallel sailing. Suppose that a ship sails due east or west from A to B. Then, as in Figure 14, the track of the ship is along the parallel of latitude through A and B. Let λ be their latitude and d be the distance AB *measured along the* λ-*latitude circle.* Then, for the path AB, the *difference in latitude* is *zero* and the *departure* is the distance d. The difference of the longitudes of A and B is the angular measure of arc AB on the λ-latitude circle. Since the length of $1'$ of arc on this circle is $\cos\lambda$ *miles,* the angular measure, DLo, of AB *in minutes* is given by the equation

$$DLo = \frac{d}{\cos\lambda}. \tag{1}$$

Since $d = dep$ in this case, from (1) we obtain

$$(\textbf{\textit{DLo}}\ \textbf{in minutes}) \qquad DLo = \frac{dep}{\cos\lambda}. \tag{2}$$

Or,

$$DLo = (dep)(\sec\lambda). \tag{3}$$

Example 1. An airplane flies from A: $(52°\ 16'\ N,\ 121°\ 32'\ W)$ due west for 250 miles to B. Find the latitude and longitude of B.

Solution. 1. In the data, the latitude of A is given first. The airplane does parallel flying; hence, the latitude of B is $52°\ 16'\ N$.

2. The departure is 250 miles. From (3),

$$DLo = 250 \sec 52°\ 16' = 408.5'\ (west). \qquad \text{(Table XI)}$$

Hence, the longitude of B is $(121°\ 32' + 408.5')$ or $128°\ 20.5'\ W$.

Example 2. If an airplane flies from A: $(36°\ N,\ 48°\ E)$ to B: $(36°\ N,\ 51°\ E)$ along the 36°-latitude circle, find the departure.

Solution. $DLo = 3° = 180'$. Hence, from (2),

$$dep = 180 \cos 36° = 145.6\ \textit{nautical miles}.$$

267. Middle latitude sailing. The formula of the preceding section relating departure and difference in longitude *does not apply* if a ship follows a track AB which is *not* all on a parallel of latitude. In this case, let L_A and L_B be the latitudes of A and B and let L_m be the average of L_A and L_B. We shall call L_m the **middle latitude** of the path:

$$L_m = \tfrac{1}{2}(L_A + L_B). \tag{1}$$

Then, for tracks AB of moderate length, useful results are arrived at by the following method. When the following formulas are used, we say that navigation is being done by **middle latitude sailing.**

I. *If β is the acute angle between the track AB and the meridian of A, use* **plane sailing** *formulas to relate the DL and dep, in Figure* 13:

$$dep = d \sin \beta; \qquad DL = d \cos \beta \ (\textit{miles or minutes}). \tag{2}$$

II. *Convert the departure to difference in longitude, DLo, by acting as if the departure occurred on the parallel of* **latitude midway between *A* and *B*.** *That is, use formula* (2) *of the preceding section with L_m as the latitude:*

$$(DLo \text{ in minutes}) \qquad DLo = \frac{dep}{\cos L_n}. \tag{3}$$

Example 1. A ship sails for 105 miles on course 307° 25′ from point A: (27° 38′ N; 56° 53′ W). Find the latitude and longitude of the destination B by middle latitude sailing.

Solution. 1. From right $\triangle ABH$ in Figure 13, Section 263,

$$dep = 105 \sin 52° \, 35' = 83.4 \text{ miles};$$
$$DL = 105 \cos 52° \, 35' = 63.8 \text{ miles};$$
$$DL = 63.8' = 1° \, 3.8' \textit{ in angular measure.}$$

2. Since A has *north* latitude and DL is to the *north,*

$$(\textit{latitude of } B) = 27° \, 38' + 1° \, 3.8' = 28° \, 42' \, N.$$

3. The middle latitude is

$$L_m = \tfrac{1}{2}(27° \, 38' + 28° \, 42') = 28° \, 10'.$$

Hence, from (3),

$$DLo = \frac{83.4}{\cos 28° \, 10'} = 94.6' = 1° \, 34.6' \qquad \text{(Logarithms)}$$

Since the departure is to the *west* and A has *west longitude,*

$$(\textit{longitude of } B) = 56° \, 53' + 1° \, 35' = 58° \, 28'.$$

To find the course and distance between two given points, we first use (3) to obtain the departure and then finish the solution by the plane sailing method.

Example 2. An airplane is to fly from the point A: (25° 36′ N, 110° 56′ W) to the point B: (28° 14′ N, 115° 16′ W). Find the course and distance by middle latitude sailing.

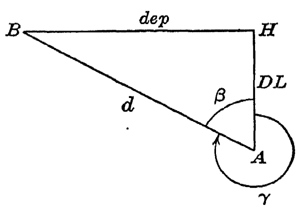


FIG. 15

Solution. 1. $DL = 2° \, 38' = 158'$ 158 miles, shown in Figure 15.

2. $L_m = 26° \, 55'$. DLo $4° \, 20' = 260'$ (*west*).

3. From formula (3),

$$dep = 260 \cos 26° \, 55' = 232 \text{ miles}.$$

4. In Figure 15, we solve right $\triangle ABH$ for β and d:

$$\tan\beta = \frac{232}{158};\quad \beta = 55^\circ\,43';\quad d = \frac{dep}{\sin\beta} = 281 \text{ miles.}$$

The course is $\gamma = 360^\circ - 55^\circ\,43' = 304^\circ\,17'$; the distance from A to B is 281 miles. (Five-place tables were used.)

268. Dead reckoning. At any instant, the navigator of a ship will have information about any ocean current which is present and the speed of the ship with respect to the water. Similarly, the navigator of an airplane will have information about the airspeed of the airplane and the wind velocity. With such data, the resultant course and groundspeed for any specified compass heading can be found by combining velocity vectors. Suppose, then, that the navigator determines a course, a distance, or his latitude and longitude at any instant by plane sailing or middle latitude sailing,* aided by the available information about his groundspeed. Such actions are referred to as navigation by **dead reckoning.** For a short journey, the errors resulting from such approximate methods may be negligible. Also, the results of dead reckoning are sometimes indispensable as preliminary data in obtaining accurate results by use of astronomical methods, which we shall mention later.

NOTE 1. Although some of the operations involved in dead reckoning have been introduced in this text as trigonometric problems, it should be noted that, in practice, the operations frequently can be carried out with sufficient accuracy by graphical means.

EXERCISE 11

Use middle latitude or parallel sailing in all problems. Obtain angular measures to the nearest minute. Signs attached to dep and DL indicate directions as specified in Section 263. Use five-place tables.

With the given departure and difference in latitude of B with respect to A, find the latitude and longitude of B. The latitude and longitude of A are given in parentheses.

1. A: $(23^\circ\,15'\,N,\ 83^\circ\,15'\,W)$; dep $= +\,135$ mi.; $DL = 0$.

2. A: $(53^\circ\,16'\,N,\ 12^\circ\,47'\,E)$; dep $= -\,240$ mi.; $DL = 0$.

3. A: $(35^\circ\,38'\,N,\ 64^\circ\,25'\,W)$; dep $= +\,240$ mi.; $DL = +\,320$ miles.

4. A: $(46^\circ\,24'\,N,\ 78^\circ\,43'\,E)$; dep $= -\,180$ mi.; $DL = +\,260$ miles.

5. A: $(15^\circ\,45'\,S,\ 158^\circ\,26'\,E)$; dep $= +\,156$ mi.; $DL = -\,250$ miles.

6. A: $(13^\circ\,24'\,S,\ 146^\circ\,50'\,W)$; dep $= -\,250$ mi.; $DL = -\,150$ miles.

* Or, other approximate methods not involving astronomical observations.

Find the latitude and longitude of an airplane at the end of $1\frac{1}{2}$ hours if it flies from A with the specified course and groundspeed, per hour.

7. A: $(62° 38' N, 126° 40' W)$; course 285°; groundspeed = 240 miles.
8. A: $(36° 15' N, 82° 25' E)$; course 130°; groundspeed = 280 miles.
9. A: $(23° 38' S, 145° 40' E)$; course 85°; groundspeed = 300 miles.
10. A: $(15° 45' S, 136° 50' E)$; course 250°; groundspeed = 225 miles.

Find the course and distance for an airplane flying from the first place to the second.

11. A: $(26° 32' N, 52° 15' W)$; B: $(28° 48' N, 54° 26' W)$.

12. A: $(35° 46' N, 126° 38' W)$; B: $(34° 2' N, 128° 54' W)$.

13. Liverpool $(53° 24' N, 3° 4' W)$; Berlin $(52° 32' N, 13° 25' E)$.

14. Moscow $(55° 45' N, 37° 37' E)$; Berlin.

15. Moscow; Paris $(48° 50' N, 2° 20' E)$.

★16. An airplane is to fly with an airspeed of 300 miles per hour from Honolulu $(21° 20' N, 157° 50' W)$ to Midway Island $(28° 10' N, 177° 20' W)$. A wind of 50 miles per hour is blowing from 240°. Find the heading which the navigator should take and the length of the path.

HINT. First find the course by middle latitude sailing and then the heading by adding vectors. Assume that speeds are in nautical miles.

269. Great circle sailing. The path of shortest distance between two points A and B on the earth's surface is the arc AB, at most 180°, of the *great circle* through A and B. This shortest distance is called the **great circle distance** between A and B. In great circle sailing from A to B, the path or track of the ship (or airplane) is the arc of the great circle from A to B. A fundamental problem of navigation is the determination of the great circle distance from A to B when the latitudes and longitudes of the points are known, and the direction of the great circle track at any point between A and B.

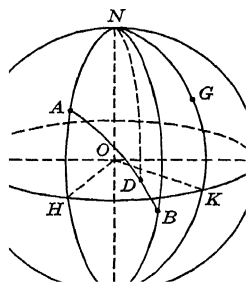


FIG. 16

Illustration 1. In Figure 16, the course of AB at A is about 130°, and is called the *initial course* of the path AB. If D is any point on AB, and ND is an arc of the longitude circle through D, the course of AB at D is $\angle NDB$. The course changes continually as one goes along AB from A to B.

Many problems of great circle navigation are solved by use of the spherical triangle NAB, in Figure 16. If G represents Greenwich, $\angle GNA$ and $\angle GNB$ are the longitudes of A and B; $\angle BNA$ is found by subtraction in the proper order as soon as the longitudes of A and B are given. In Figure 16,

$$AN = \textit{colatitude of } A;$$

$$BN = 90° + (\textit{latitude of } B),$$

since B is south of the equator. The angular measure of a great circle arc AB can be translated into linear distance by recalling that *1′ of arc on a great circle is 1 nautical mile in length.*

Example 1. Find the great circle distance from New York (40° 49′ N, 73° 58′ W) to Rio de Janeiro (22° 54′ S; 43° 10′ W) and the initial course of the track.

Solution. 1. In Figure 16, let A and B represent the cities and let us think of N as C in the standard triangle of Chapter XVI. Then, in $\triangle ABN$, where $\angle ANB = \gamma$,

$$\gamma = 73° 58' - 43° 10'; \quad b = AN = 90° - 40° 49';$$
$$a = NB = 90° + 22° 54'.$$

Or, $a = 112° 54'$; $b = 49° 11'$; $\gamma = 30° 48'$. The solution of $\triangle ABN$ can be obtained by cutting it into two right triangles, by constructing a great circle through A perpendicular to BN, as in Exercise 3. Or, the triangle can be solved by the method * of Case III, Section 255. By use of four-place logarithms, we obtain $c = AB = 69° 51'$, $\alpha = 149° 50'$; and $\beta = \angle ABN = 24° 23'$.

2. The initial course is 149° 50′. The bearing of New York from Rio de Janeiro is 335° 37′, or 24° 23′ west of north.

3. To find the linear distance AB:

$$69° 51' = 4191'; \qquad AB = 4191 \textit{ nautical miles;}$$
$$AB = 4191(1.1515) = 4826 \textit{ statute miles.}$$

NOTE 1. After solving Example 1, for any point D on AB which has a specified latitude we could find the longitude of D and also the course, or direction of the track at D, by solving the spherical triangle with vertices B, N, and D. Or, if the longitude of D were specified we could find its latitude. In this way we could compute as many points as desired along AB and the directions of the track at these points.

* If just the great circle distance is desired, a formula of the law of cosines for sides (Section 249) is convenient. Also, for this purpose, the haversine modification of the law of cosines is convenient if a table of haversines is available.

EXERCISE 12

Find the great circle distance in nautical miles between the cities and the initial course of the great circle track from the first city to the second. Also, find the bearing of the second city from the first. Use five-place tables.

1. New York (40° 49′ *N*, 73° 58′ *W*); Paris (48° 50′ *N*, 2° 20′ *E*).
2. Los Angeles (34° 3′ *N*, 118° 15′ *W*); Manila (14° 35′ *N*, 120° 59′ *E*).
3. Los Angeles; Tokyo (35° 39′ *N*, 139° 45′ *E*).
4. Honolulu (21° 18′ *N*, 157° 52′ *W*); Tokyo.
5. Seattle (47° 40′ *N*, 122° 19′ *W*); Tokyo.
6. Manila; Tokyo.
7. Los Angeles; Honolulu.
8. Gibraltar (36° 6′ *N*, 5° 21′ *W*); Rio de Janeiro (22° 54′ *S*, 43° 10′ *W*).
9. Sydney (33° 52′ *S*, 151° 12′ *E*); San Francisco (37° 47′ *N*, 122° 26′ *W*).
10. Liverpool (53° 24′ *N*, 3° 4′ *W*); Hongkong (22° 18′ *N*, 114° 10′ *E*).
11. San Francisco; New York.
12. New York; Los Angeles.
13. Moscow (55° 45′ *N*, 37° 37′ *E*); Paris.
14. Dakar (14° 40′ *N*, 17° 26′ *W*); Rio de Janeiro.
15. Moscow; Tokyo.
16. Liverpool; Berlin (52° 32′ *N*, 13° 25′ *E*).
17. Tokyo; Dutch Harbor, Alaska (53° 48′ *N*, 166° 25′ *W*).
18. Midway Island (28° 13′ *N*, 177° 23′ *W*); Dutch Harbor.
19. Minneapolis (44° 59′ *N*, 93° 17′ *W*); Dutch Harbor.
20. Minneapolis; Moscow.
21. New York; Moscow.
22. Midway Island; Honolulu.

Find the difference, in statute miles, between the great circle distance and the distance along a parallel of latitude, between the two cities.

23. New York (40° 50′ *N*, 74° 0′ *W*); Salt Lake City (40° 50′ *N*, 111° 50′ *W*).
24. San Diego (32° 40′ *N*, 117° 10′ *W*); Charleston (32° 40′ *N*, 79° 50′ *W*).

270. The celestial sphere. On observing the sky, we see the heavenly bodies as though they were on a *sphere* with our observation point as the center. For our purposes, this **celestial sphere** may be thought of as so large that, by comparison, the earth's radius is of negligible length, and hence we may refer to the earth as the center of the sphere. The **celestial north pole** P and **south pole** P' are the points where the earth's axis, extended through the earth's north and south poles, respectively, meets the celestial sphere. To any observer, this sphere appears to rotate from *east to west* about the axis PP', because the earth rotates from *west to east* about its axis.

NOTE 1. Any great circle referred to will be on the celestial sphere.

By the *position* of a celestial object, we shall mean the point where the line of sight from an observer to the object meets the celestial sphere. To discuss the positions of celestial objects, we shall introduce the following astronomical terms, which are illustrated in Figure 17.

1. The **celestial equator** is the great circle whose poles are P and P' and is the intersection of the plane of the earth's equator and the celestial sphere.

2. At any instant, the **zenith** of an observer is the point Z vertically above him on the celestial sphere; the point Z' diametrically opposite to Z is called the **nadir.**

3. The **horizon** of an observer is the great circle whose poles are Z and Z'. The plane of the horizon is tangent to the earth's surface at the observer's position.

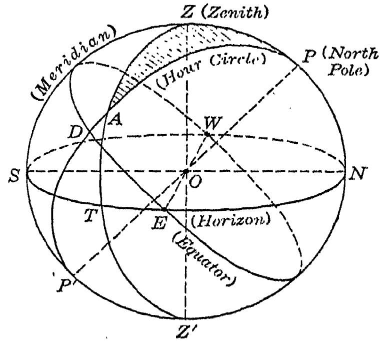


FIG. 17

4. The **celestial meridian** is the great circle through the zenith Z and the north pole P. The intersections N and S of this meridian and the horizon are called the **north** and **south points,** respectively, where N is the intersection *nearest* to P. If the observer faces N, the points E and W to his right and left on the horizon are called the **east** and **west points,** respectively. The equator and the horizon intersect at E and W.

5. The **hour circle** of a point A on the celestial sphere is the great circle through P and A.

6. The **hour angle** * of A is the angle ZPA between the meridian and the hour circle of A, measured east or west from the meridian from 0° to 180°. This hour angle is measured in *hours* or in degrees, where *one hour equals* 15°. One *minute* in the hour system is $\frac{1}{60}$th of 15° or 15′ in usual measure; one *second* in the hour system is 15″. In this book, to avoid confusion with former notations for angular measure, the words *minutes* and *seconds* will usually be written and

* More properly, the *local* hour angle, as distinguished from the Greenwich and sidereal hour angles.

not be *abbreviated* by minute or second *signs* in dealing with the hour system for measuring hour angles.

NOTE 1. The hour system of measurement proves useful because the celestial sphere rotates through 15° in one *sidereal hour*, which is about 10″ less than an ordinary *solar hour*.

7. The **declination** of A is the angular distance of A from the equator, measured along the *hour circle* of A. The declination, DA in Figure 17, is designated as *positive* or *negative* according as A is *north* or *south* of the equator.

8. The **vertical circle** of A is the great circle through Z and A.

9. The **altitude** of A is the angular distance of A from the horizon, measured along the vertical circle of A. The altitude, TA in Figure 17, is designated as *positive* or *negative* according as A is *above* or *below* the horizon.

10. The **azimuth** of A is the angle NZA between the meridian and the vertical circle of A, counted from 0° to 360° east (clockwise) from the north point.* In Figure 17, arc NET measures the azimuth of A.

We notice that the altitude, azimuth, and hour angle of A depend not only on the position of A on the celestial sphere but also on the *time of observation* and the *position of the observer* on the earth. The declination depends *only* on the *position on the celestial sphere.*

At any instant, the position of point A on the celestial sphere is uniquely determined with respect to an observer if the *altitude* and *azimuth* are known. This means of locating A can be referred to as the **altitude-azimuth coordinate system.** Also, the position of A can be described by specifying its *hour angle* and *declination;* this means for designating the position of A is called the **declination-hour-angle system of coordinates.**† Either system corresponds in its essential nature to the latitude-longitude system for indicating positions on the earth's surface.

* Infrequently, $\angle SZA$, measured clockwise through W, is defined as the azimuth.

† Another coordinate (see texts on astronomy and navigation) called the **right ascension** of A is paired with the declination to give a third system of coordinates on the celestial sphere. Tables giving the Greenwich hour angles of celestial objects make it unnecessary to use right ascensions in many important types of problems. However, the **declination-(right-ascension)** system of coordinates is very important in astronomy and navigation.

271. The altitude of the north pole of the celestial sphere is the same as the latitude of the observer.* To see this, consider the plane section of the earth in the figure where O is the observer, T is the earth's north pole, and T' is the earth's south pole. The student should imagine himself standing with his feet at O and his head in the direction OZ. The arrows OZ, OP, and ON point at the zenith, north pole, and north point, respectively, on the celestial sphere. We verify that θ is the altitude of P. In Figure 18, we notice that ON and OP are perpendicular, respectively, to the sides of angle λ. Hence $\theta = \lambda$. But, λ is the angle whose measure is the latitude of O.

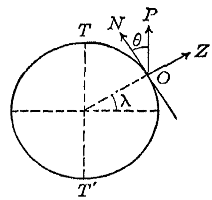


FIG. 18

272. The astronomical triangle for the point A at any instant is the spherical triangle ZPA of Figure 17, where

$$
\begin{aligned}
ZP &= 90° - (\textit{altitude of } P) = \textit{colatitude of } O;\\
AZ &= 90° - (\textit{altitude of } A) = \textit{coaltitude of } A;\\
AP &= 90° - (\textit{declination of } A) = \textit{codeclination of } A;\\
\angle ZPA &= \textit{hour angle of } A \ (\textit{east or west of meridian});
\end{aligned}
$$

$$
\angle PZA \begin{cases} = \textit{azimuth (if } A \textit{ is east of meridian)};\\ = 360° - \textit{azimuth (if } A \textit{ is west of meridian)}. \end{cases}
$$

NOTE 1. Problems associated with the astronomical triangle are of fundamental importance in navigation as well as in certain problems which arise in astronomy. It should not be assumed, however, that a navigator frequently carries out the trigonometric solution of spherical triangles as required in the exercises. Excellent existing navigation tables and special computing instruments enable him to eliminate many extensive computations which would otherwise be involved in the solution of various astronomical triangles.

NOTE 2. The declination of the sun, certain planets, and 55 prominent stars at all times during each year are available in American navigation tables. Also, when an observer's *longitude* is known, these tables give him the (*local*) *hour angle* of any one of these celestial bodies at any time.

For any observer, **solar noon** is the instant when the sun is on arc PZS of the meridian. The *local solar time* at any moment is speci-

* In Figure 18, associated remarks, and certain later formulas involving the pole P, we assume that the observer is in the northern hemisphere. Similar details relating to pole P' would apply for an observer in the southern hemisphere.

fied as before or after solar noon. If A, in Figure 17, represents the sun, at t hours before noon $\angle ZPA = t$ hours; at t hours after noon, A is on the western side of the celestial sphere and again $\angle ZPA = t$ hours.

I.* *To find the solar time of an observer, whose latitude is known, at an instant when the sun's declination and altitude are given.*

Example * 1. Find the solar time in New York (latitude: 40° 50′ N) at a moment in an afternoon when the sun's declination is 16° 30′, if the sun's altitude is observed to be 35° 20′.

Solution. 1. In triangle ZPA, where A is the sun, we know all the sides: $ZP = 49° 10'$; $ZA = 54° 40'$; $AP = 73° 30'$.

2. For temporary convenience, let $a = 49° 10'$; $b = 73° 30'$; $c = 54° 40'$; $\angle ZPA = \gamma$. By use of four-place logarithms, from either one of (XII) in Section 250, or from (IV) in Section 249, we obtain $\gamma = 57° 14'$.

3. We reduce 57° 14′ to the astronomical angular units *hours* and *minutes*, where *one hour* = 15°; *one minute* = 15′. We find that 57° 14′ is equal to 3 *hours* and 49 *minutes:* $\gamma = 3^{hr.} 49'$. Since the celestial sphere turns through 15° in one hour † of *time*, the observer took his observation at 3:49 P.M.

II. *To find the solar time of sunrise (or sunset) in a given latitude when the declination of the sun is known.*

Problem II is a special case of Problem I where the sun's altitude is 0°. Triangle ZPA has $ZA = 90°$, so that the triangle is quadrantal and can be solved by the methods of Chapter XV.

III. *To find the latitude of the observer if the altitude, hour angle, and declination of a celestial object are known.*

Example 2. At 3:37 P.M. local solar time, when the sun's declination is − 16° 12′, an observer finds that the sun's altitude is 22° 18′. Find his latitude.

Solution. In triangle ZPA, where A is the sun, $ZA = 67° 42'$; $PA = 106° 12'$; $\angle ZPA = 3$ hours and 37 minutes west; or, $\angle ZPA = 54° 15'$. On solving the triangle we would obtain ZP; the latitude is $(90° - ZP)$.

IV. *When the observer's latitude and longitude ‡ are given, to find the altitude and azimuth of a celestial body whose declination and hour angle are known.*

* Should be omitted by students who have not studied Chapter XVI.

† We disregard the small difference between a solar and a sidereal hour.

‡ See Note 2 for the practical necessity of knowing the longitude. In problems of type (IV) in this book, the longitude usually will not be given because no navigation tables are to be employed.

Example 3. At latitude (44° 10′ N, longitude 99° 50′ W), at a certain instant a star A has declination 22° 57′ and hour angle 3 hours, 12 minutes east. Find the altitude and azimuth of A.

Solution. In $\triangle PZA$ in Figure 19,

$PZ = 45° 50'$; $PA = 67° 3'$; $\angle ZPA = 48° 0'$.

We desire to find the coaltitude ZA and the azimuth $\angle PZA$. The solution comes under Section 255, or can be accomplished as in the following comment. We obtain, finally,

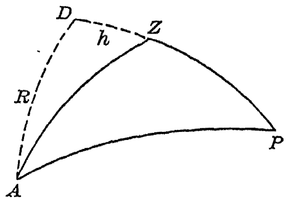


FIG. 19

altitude $= 45° 32'$; *azimuth* $= 102° 20'$. (1)

Comment. In the well-known **Ageton method** for solving $\triangle PZA$ in navigation, the solution in Example 3 is obtained by first passing a great circle through the star A perpendicular to PZ. This gives right triangles PAD and ZAD, in Figure 19, whose solution in succession gives (1). In this method, a suitable sequence of formulas, in the order used, as obtained by Napier's rules and geometry are as follows:

From $\triangle PDA$ *to find* R: $\sin R = \sin PA \sin \angle ZPA.$ (2)

From $\triangle PDA$ *to find* PD: $\cos PA = \cos R \cos PD.$ (3)

From $\triangle DZA$, *with* $h = PD - PZ$, $\cos ZA = \cos h \cos R.$ (4)

From $\triangle DZA$, *to find* $\angle DZA$: $\sin R = \sin \angle DZA \cdot \sin ZA.$ (5)

From the figure: $\angle PZA = 180° - \angle DZA.$ (6)

In navigation, (2) to (5) are expressed entirely in terms of secants and cosecants of the declination, altitude, hour angle, the complement of PD, the azimuth, and R, to facilitate the use of associated tables which give a convenient multiple of the logarithms of secants and cosecants.

EXERCISE 13

1.* Find the solar time in Chicago (latitude: 41° 50′ N) at a moment in a morning when the sun's declination is 14° 16′ if the sun's altitude is observed to be 38° 14′.

2.* Find the solar time in Washington, D.C. (latitude: 38° 55′ N) at a moment in an afternoon when the sun's declination is 18° 20′ if the sun's altitude is observed to be 49° 36′.

3. Find the Chicago time of sunrise on the morning of Problem 1, and the bearing of the sun at sunrise.

4. Find the Washington time of sunset on the afternoon of Problem 2, and the bearing of the sun at sunset.

* These problems should be omitted by students who have not studied Chapter XVI.

NOTE. The sun's declination is $23° 27'$ on the longest day of the year and $-23° 27'$ on the shortest day of the year, for the northern hemisphere of the earth. We shall assume that all problems refer to observers in this hemisphere. To find the length of one of these days, we find the solar time at sunrise on that day.

Find the lengths of the longest and the shortest days in each city and the bearing of the sun at sunrise on each of these days.

5. Fairbanks, Alaska (latitude: $64° 51' N$).

6. New Orleans ($29° 57' N$).

7. At 9:28 A.M., local solar time, an observer finds that the sun's altitude is $37° 26'$, on a day when the sun's declination is $6° 20'$. Find the observer's latitude.

8. At 2:36 P.M., local solar time, an observer finds that the sun's altitude is $47° 32'$, when the sun's declination is $-3° 16'$. Find the observer's latitude.

9. At a moment when the sun's declination is $11° 16'$, an observer finds that the sun's azimuth is $311° 26'$ and its altitude is $38° 22'$. Find the observer's latitude.

10. At 9:46 A.M., local solar time in Chicago (latitude: $41° 50' N$), the sun's altitude is observed to be $19° 20'$. Find the sun's declination then.

Find the altitude and azimuth of a star A as observed from a point D whose latitude is λ at an instant of time when d is the declination and H is the local hour angle of A.

11. $\lambda = 53° 20' N$; $d = 28° 35'$; $H = 2$ hours, 17 minutes, east.

12. $\lambda = 35° 16' N$; $d = 48° 17'$; $H = 3$ hours, 25 minutes, east.

13. $\lambda = 22° 25' N$; $d = 76° 25'$; $H = 4$ hours, 40 minutes, west.

14. $\lambda = 17° 40' N$; $d = -12° 50'$; $H = 4$ hours, 22 minutes, west.

15. $\lambda = 35° 26' S$; $d = -25° 30'$; $H = 2$ hours, 10 minutes, east.

273. Remarks on celestial navigation.* When astronomical observations and related computations are used in navigating a ship (or airplane), we say that **celestial navigation** is being employed. Two important duties of the navigator are (1) to determine the latitude and longitude of his position frequently and then (2) to redirect the course of the ship if necessary. Action (2) can be performed very simply after action (1) has been carried out. In a certain important method for finding the position of a ship at a given instant, Case IV of Section 272 is an essential feature. Before

* This section is merely descriptive; it is designed to show the practical significance of some of the problems in the preceding exercise.

outlining this method let us consider the following geometrical background.

First let us recall a property of latitude circles. If the star POLARIS is assumed to be exactly * at the north pole P of the celestial sphere, a direction line from the earth's center O to Polaris pierces the surface exactly at the earth's north pole N; an observer there would see Polaris at the zenith. Now suppose that the observer is at some other point D and finds that the altitude of Polaris is λ. Then he knows, from Section 271, that D is on a certain parallel of latitude which has N as a pole, and that the angular measure of the great circle distance DN is $(90° - \lambda)$; if $(90° - \lambda)$ is expressed in minutes then D is $(90° - \lambda)$ miles from N.

Now consider any star A. At a specified instant of time t, a direction line from the center of the earth to A pierces the surface at some point † A'. Then, just as there are latitude circles related to Polaris, there are small circles, which we shall call **position circles,** related to A. If an observer at D on the earth finds that the altitude of A is δ, then D lies on a position circle C_A with A' as a *pole* and the angular measure of the great circle distance DA' is $(90° - \delta)$.

Suppose, now, that an observer is at D at some instant of time t and desires to find his latitude and longitude, but knows merely the *general region* in which D is located. If he observes the altitude of a certain star A to be δ_A, then D must be on a position circle C_A at a great circle angular distance $(90° - \delta_A)$ from A'. If he also observes the altitude of a second star B to be δ_B, then D must be on a second position circle C_B at the distance $(90° - \delta_B)$ from a corresponding point B'. If a sphere were available on which he could plot A' and B' and draw C_A and C_B, as in Figure 20, he could now locate D graphically. For, D is that one of the two points of intersection of C_A and C_B which lies in the general region where the observer knows that he is located. (Usually, the other point of intersection of C_A

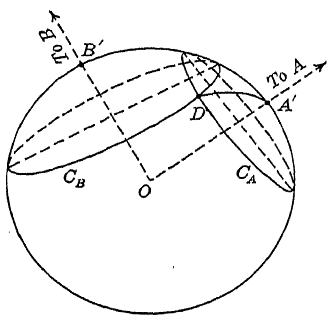


FIG. 20

* Actually, the great circle distance from Polaris to P is about 1°.

† Due to the rotation of the earth, the location of the **sub-astral** point A changes as t changes.

and C_B would be an obviously impossible location for D.) Thus, *observation of the altitudes of two stars determines the position of D.*

The use of circles C_A and C_B on a sphere would be impractical for actual navigation, unless elaborate apparatus were available, but a very useful method of position finding is based on the preceding notions. The method involves the construction, on a chart, of straight line approximations to small arcs of C_A and C_B intersecting at D. The typical sequence of actions which the navigator takes in this method are as follows, where it is assumed that he knew his position at some *previous* time t_1 and desires to find his latitude and longitude *now*, at time t_2.

1. From knowledge of the position at time t_1, the course on which the ship was then headed, and any other information, the navigator makes an assumption as to the position of the ship at the present time t_2. Let the assumed * position D' have latitude λ and longitude ϕ. Let the actual position be D.

2. With a sextant, he measures the altitudes δ_A and δ_B of two stars A and B, respectively.

3. From his tables he reads the declination and hour angle which A would have at time t_2 if the ship were at D'. Then, he solves † a problem of Type IV of Section 272 to obtain the altitude δ'_A and azimuth θ'_A which A would have at time t_2 if the ship were at D'.

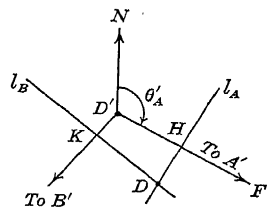


FIG. 21

4. On his chart, as in Figure 21, the navigator constructs straight line $D'F$ at azimuth θ'_A. $D'F$ is a chart approximation to a piece of an arc like DA' in Figure 20 and is assumed to be directed at A'. Suppose that $\delta'_A < \delta_A$; then D is closer than D' to A' because the observed altitude of A *increases* if we approach ‡ A' on the earth. If DA' and $D'A'$ are the great circle distances, in angular measure, from D and D' to A', then

$$DA' = 90° - \delta_A; \quad D'A' = 90° - \delta'_A; \quad D'A' - DA' = \delta_A - \delta'_A.$$

If $(\delta_A - \delta'_A)$ is measured in minutes, it is the value of $(D'A' - DA')$

* D' is frequently taken as a convenient point near the position estimated by methods of dead reckoning but this is not necessarily the case.

† In practice, he may be able to read the results directly from tables.

‡ The altitude of A is 90° as seen from A'.

in nautical miles. Hence, the position circle C_A, on which D lies, cuts $D'F$ at a point H where $D'H = (\delta_A - \delta'_A)$ nautical miles *in the direction* of A'. The navigator locates H by measurement on the chart and draws a line segment l_A perpendicular * to $D'F$ as an approximation to an arc of C_A. We call l_A a **position line**; if D is reasonably close to D', no appreciable error is made if it is assumed that D is on l_A. If $\delta_A < \delta'_A$, then l_A would be drawn perpendicular to $D'F$ at a point H in the direction *away* from A' so that we would have $D'H = (\delta'_A - \delta_A)$ nautical miles. Line l_A is called a **Sumner line.**

5. Steps (3) and (4) are repeated for star B and the navigator obtains a second position line l_B on which D lies, approximately. Then, D is located as the intersection of l_A and l_B, as in Figure 21. The process of locating D is called **making a fix.**

* An arc of C_A near to H is approximated closely by the tangent to C_A at H, and this tangent is perpendicular to the direction of $D'F$.

ANSWERS TO EXERCISES

PART II. SOLID GEOMETRY

Page 152

1. (*a*) Yes. (*b*) Yes.
3. It cuts through the surface.
5. (*a*) Yes. (*b*) Yes.
7. All lie on the plane.
9. (*a*) Yes. (*b*) An infinite number of planes can contain 2 points in space.
11. No.

Page 153

1. Because 3 points determine a plane.
3. No; they may be skew lines.
5. In one point.

Page 154

3. (*a*) Yes. (*b*) Yes.
5. (*b*) One and only one.
7. (*a*) Yes. (*b*) No. (*c*) One and only one line passing through a point outside a plane can be $\perp$ to the plane.

Page 157

3. Equal.

Page 158

1. (*b*) Yes.
3. (*b*) Two planes with no points in common are parallel.
5. (*b*) Parallel lines.
7. It is parallel to the plane.
9. (*b*) AB is parallel to plane RS.
11. An infinite number.

Page 162

1. They are parallel to the given line.
3. The intersections are parallel lines.

Page 167

1. (*a*) Yes. (*b*) No.

Page 169

1. (*a*) 8.66 in. (*b*) 7.07 in. (*c*) 5.0 in.
3. (*a*) 6.428 in. (*b*) 9.642 in. (*c*) 12.856 in.
5. 19.15 in.
7. Its projection is a point.
15. Yes, or a straight line.

Page 170

1. $AB \parallel CD$.
3. $AB \parallel CD$.
5. $RS \parallel MN$.
7. X is in plane MN.
9. (*a*) $XY \perp$ plane RS. (*b*) XY lies in MN. (*c*) $\angle XYZ$ is the plane angle of the dihedral angle formed by MN and RS. It is a right angle.

11. $\angle XOZ < 100°$; $\angle XOZ > 20°$. **13.** X is in plane RS.
15. $AB \perp$ plane RS. **17.** $AC \perp$ plane XY. **19.** $AB \parallel CD$.
21. $AB \perp$ plane XYZ. **23.** Planes ABX and CDY are perpendicular to MN.
25. The intersections of MN and PQ with the dihedral angle will be equal angles.

Page 173

1. (*a*) 62.8 in. (*b*) 15.7 in. **3.** 10.47 in.

Page 174

1. 5.2 in. **3.** 6.3 in. **5.** 18.8 in.

Page 175

1. A line or plane tangent to a sphere is perpendicular to the radius of the sphere drawn to the point of tangency. **3.** (*a*) An infinite number.

Page 176

1. (*a*) 60°. (*b*) 90°.

Page 178

5. 2 hr. **7.** 480 naut. mi. **9.** (*b*) 21,600 cos L naut. mi.
11. (*a*) Midnight. (*b*) 10,800 naut. mi.

Page 179

1. Less than 62.8 in. **3.** (*a*) $\frac{1}{2}$. (*b*) $\frac{3}{4}$. (*c*) $\frac{5}{6}$.

Page 181

1. 103°; 57°; 85°. **3.** Equiangular.

Page 182

3. (*a*) 90°. (*b*) 60°. (*c*) 210°.
5. (*a*) Two are right angles; the third is an acute angle.
(*b*) Two are right angles; the third is an obtuse angle.
7. (*a*) The opposite vertex. (*b*) Have the same measure (90°).
(*c*) 5400 naut. mi. (*d*) 98.9 in.

Page 189

1. 314 sq. in. **3.** 628 sq. in. **5.** $\frac{1}{8}$. **7.** 201,000,000 sq. mi.
9. $\frac{1}{48}$ the surface of the earth. **11.** $2.09r^2$.

Page 194

1. (*a*) No. (*b*) Yes. (*c*) No. **3.** 45°.
5. No; the parallel of latitude is not a great circle. **7.** 1256 sq. in.
9. (*a*) 157 sq. in. (*b*) 471 sq. in. (*c*) 216 sq. in. (*d*) 157 sq. in.
11. (*a*) Three equal face angles. (*b*) Two equal face angles.
(*c*) Three equal dihedral angles. (*d*) Two equal dihedral angles.
13. 1.8 times it. **15.** (*a*) $\frac{9}{16}$. (*b*) $\frac{27}{64}$. **17.** 50,240,000 sq. mi.

Page 196

(*a*) 27.75 sq. ft. (*b*) 340 sq. cm. (*c*) 38.125 sq. ft. (*d*) 7 sq. yd.
(*a*) 89.25 sq. cm. (*b*) 8.95 in. (*c*) 11.5 ft. **5.** A rectangle.

Page 197

420 lb. **3.** $33\frac{1}{3}$ cu. yd. **5.** 967.7 lb., or 968 lb. **7.** 840 lb.
(*a*) Multiplied by 4. (*b*) Multiplied by 8.

Page 198

(*a*) 288 cu. in. (*b*) 70.146 cu. in., or 70.1 cu. in.
(*c*) 748.224 cu. in., or 748 cu. in.

Page 199

$33\frac{1}{3}$ cu. yd. **3.** $333\frac{1}{3}$ cu. yd.
(*a*) $3\frac{2}{3}$ cu. ft. (*b*) 605 lb.; $2038\frac{2}{3}$ lb.

Page 201

324 sq. in. **3.** (*a*) 2.5 sq. in. (*b*) $4\frac{4}{9}$ sq. in. (*c*) $15\frac{5}{8}$ sq. in.

Page 203

240 cu. in. **3.** One sixth of the cube.
(*b*) 15.716 in. (*c*) 282.9 sq. in. (*d*) 14.832 in. (*e*) 462.4 cu. in.
(*f*) 212.2 sq. in.

Page 204

3. 1818.6 cu. in. **5.** (*a*) $\frac{1}{4}B$. (*b*) $\frac{7}{24}hB$.

Page 205

1. 187.5 sq. in. **3.** (*a*) 7.5 in. (*b*) 12.5 in. (*c*) 140.625 sq. in.
5. 27.5 sq. ft., or 28 sq. ft. **7.** (*a*) 4 sq. ft. (*b*) 9 sq. ft. (*c*) 16 sq. ft.

Page 207

3. 5298.75 gal., or 5300 gal. **5.** 57.7 cu. in. **7.** 75.4 sq. ft.
9. 8.52 in. **11.** 1154 sq. ft.

Page 209

1. 251.2 cu. in. **3.** 12.6 cu. in.
5. (*a*) 10 in. (*b*) 157 sq. in. (*c*) 226.6 cu. in., or 227 cu. in.

Page 210

1. $4186\frac{2}{3}$ cu. in. **3.** 31,400 gal.

Page 211

1. (*a*) 7.5 sq. in. (*b*) 120 cu. in. (*c*) 15 cu. in. (*d*) 105 cu. in.
3. Volume, 201 cu. in.; lateral area, 159 sq. in.
5. (*a*) 916.2 sq. in., or 916 sq. in. (*b*) 1641.2 cu. in., or 1641 cu. in.
7. 439.6 sq. in. **9.** They are equal.

ANSWERS TO EXERCISES

PART III. SPHERICAL TRIGONOMETRY

NOTE. Results computed by use of four-place logarithms are given in heavy black type when both four-place and five-place answers are listed. Also, in most of the examples involving computation, answers will be given for both odd- and even-numbered problems. Slight variations from the listed answers may be expected in some problems involving computation because of permissible variations in the formulas employed.

Exercise 1. Page 219

3. $\cos\beta = \cot c \tan a$.

5. $\cos c = \cos a \cos b$.

7. $\cos\alpha = \sin\beta \cos a$.

9. $\cos\alpha = \cot c \tan b$.

11. $\sin a = \tan b \cot\beta$.

Exercise 2. Page 222

1. $b = \mathbf{50^\circ\,3'};\ 50^\circ\,3.0'$: $\alpha = \mathbf{52^\circ\,51'};\ 52^\circ\,50.9'$: $\beta = \mathbf{59^\circ\,13'};\ 59^\circ\,13.0'$:

2. $a = \mathbf{57^\circ\,9'};\ 57^\circ\,9.6'$: $\alpha = \mathbf{60^\circ\,17'};\ 60^\circ\,17.0'$: $\beta = \mathbf{66^\circ\,5'};\ 66^\circ\,4.7'$.

3. $a = \mathbf{58^\circ\,5'};\ 58^\circ\,4.1'$: $b = \mathbf{81^\circ\,47'};\ 81^\circ\,47.2'$: $\beta = \mathbf{83^\circ\,1'};\ 83^\circ\,1.0'$.

4. $a = \mathbf{32^\circ\,11'};\ 32^\circ\,11.0'$ $b = \mathbf{60^\circ\,40'};\ 60^\circ\,39.6'$: $\alpha = \mathbf{35^\circ\,50'};\ 35^\circ\,49.6'$.

5. $c = \mathbf{85^\circ\,37'};\ 85^\circ\,37.1'$ $\alpha = \mathbf{85^\circ\,32'};\ 85^\circ\,32.3'$: $\beta = \mathbf{45^\circ\,36'}\ \ 45^\circ\,36.2'$.

6. $c = \mathbf{85^\circ\,1'};\ 85^\circ\,1.0'$: $\alpha = \mathbf{61^\circ\,50'};\ 61^\circ\,50.1'$: $\beta = \mathbf{80^\circ\,47}\ \ 80^\circ\,47.1'$.

7. $a = \mathbf{37^\circ\,45'};\ 37^\circ\,45.0'$ $c = \mathbf{51^\circ\,38'};\ 51^\circ\,36.9'$: $\alpha = \mathbf{51^\circ\,22}\ \ 51^\circ\,21.4'$.

Or, $a = \mathbf{142^\circ\,15'};\ 142^\circ\,15.0'$: $c = \mathbf{128^\circ\,22'};\ 128^\circ\,23.1'$: $\alpha = \mathbf{128^\circ\,38'};\ 128^\circ\,38.6'$.

8. $b = \mathbf{143^\circ\,8'};\ 143^\circ\,7.9'$ $c = \mathbf{125^\circ\,30'};\ 125^\circ\,29.7'$: $\beta = \mathbf{132^\circ\,32'};\ 132^\circ\,31.8'$.

Or, $b = \mathbf{36^\circ\,52'};\ 36^\circ\,52.1$ $c = \mathbf{54^\circ\,30'};\ 54^\circ\,30.3'$: $\beta = \mathbf{47^\circ\,28'};\ 47^\circ\,28.2'$.

9. $c = \mathbf{99^\circ\,38'};\ 99^\circ\,38.4'$ $\alpha = \mathbf{62^\circ\,59'};\ 62^\circ\,58.7'$: $\beta = \mathbf{108^\circ\,10'};\ 108^\circ\,10.7'$.

10. $c = \mathbf{106^\circ\,54'};\ 106^\circ\,54.1'$ $\alpha = \mathbf{101^\circ\,20'};\ 101^\circ\,20.3'$: $\beta = \mathbf{59^\circ\,4'};\ 59^\circ\,4.2'$.

11. $a = \mathbf{47^\circ\,31'};\ 47^\circ\,32.0'$ $b = \mathbf{104^\circ\,19'};\ 104^\circ\,19.4'$: $c = \mathbf{99^\circ\,37'};\ 99^\circ\,36.9'$.

12. $a = \mathbf{122^\circ\,18'};\ 122^\circ\,17.8'$ $b = \mathbf{67^\circ\,18'};\ 67^\circ\,18.4'$: $c = \mathbf{101^\circ\,54'};\ 101^\circ\,53.8'$.

13. No solution.

14. No solution.

15. $b = \mathbf{119^\circ\,49'};\ 119^\circ\,49.0'$ $c = \mathbf{72^\circ\,23'};\ 72^\circ\,22.8'$: $\alpha = \mathbf{123^\circ\,39'};\ 123^\circ\,39.3'$.

16. No solution.

17. $a = \mathbf{122^\circ\,28'};\ 122^\circ\,27.2'$ $b = \mathbf{163^\circ\,30'};\ 163^\circ\,28.8'$: $c = \mathbf{59^\circ\,2'};\ 59^\circ\,2.4'$.

18. $a = \mathbf{109^\circ\,54'};\ 109^\circ\,53.8'$ $c = \mathbf{97^\circ\,27'};\ 97^\circ\,27.4'$: $\beta = \mathbf{68^\circ\,48'};\ 68^\circ\,48.0'$.

19. $b = \mathbf{14^\circ\,8'};\ 14^\circ\,7.8'$: $c = \mathbf{113^\circ\,32'};\ 113^\circ\,33.0'$: $\beta = \mathbf{15^\circ\,27'};\ 15^\circ\,26.7'$.

Or, $b = \mathbf{165^\circ\,52'};\ 165^\circ\,52.2'$ $c = \mathbf{66^\circ\,28'};\ 66^\circ\,27.0'$: $\beta = \mathbf{164^\circ\,33'};\ 164^\circ\,33.3'$.

20. a = **22° 14′**; 22° 14.3′: c = **123° 11′**; 123° 10.1′: α = **26° 52′**; 26° 52.8′:
Or, a = **157° 46′**; 157° 45.7′: c = **56° 49′**; 56° 49.9′: α = **153° 8′**; 153° 7.2′.

21. $c = 90°$; $\alpha = 36°$; $\beta = 90°$. **22.** $b = 90°$; $c = 90°$; $\alpha = 90°$.

23. No solution. **24.** No solution.

25. $a = 56° 46.8'$; $b = 44° 12.8'$; $c = 66° 52.8'$.

26. $b = 69° 24.9'$; $\alpha = 37° 30.2'$; $\beta = 77° 38.4'$.

27. $b = 25° 58.0'$; $c = 139° 7.6'$; $\beta = 41° 59.8'$: or, $b = 154° 2.0'$; $c = 40° 52.4'$; $\beta = 138° 0.2'$.

28. $a = 172° 0.3'$; $c = 24° 4.3'$; $\alpha = 160° 3.8'$: or, $a = 7° 59.7'$; $c = 155° 55.7'$; $\alpha = 19° 56.2'$.

29. $a = 27° 33.3'$; $b = 134° 37.7'$; $\beta = 114° 32.6'$.

30. $a = 145° 41.3'$; $c = 131° 26.2'$; $\beta = 52° 57.4'$.

31. $a = c = 90°$; $b = \beta$. **33.** $a = \alpha = 90°$ and $b = \beta$; or $b = \beta = 90°$ and $a = \alpha$.

35. $\beta = a = c = 90°$.

Exercise 3. Page 223

1. $\alpha = \beta$ = **60° 26′**; 60° 25.5′: γ = **147° 40′**; 147° 40.6′.

2. α = **109° 24′**; 109° 24.2′: $\beta = \gamma$ = **62° 23′**; 62° 23.5′.

3. a = **72° 18′**; 72° 18.2′: $\beta = \gamma$ = **57° 12′**; 57° 12.6′.

4. $a = b$ = **53° 22′**; 53° 22.7′: c = **83° 34′**; 83° 36.2′.

5. a = **48° 48′**; 48° 47.6′: b = 37° 32′: α = **85° 22′**; 85° 22.6′.

6. α = **12° 43′**; 12° 41.7′: β = **24° 13′**; 24° 12.6′: γ = **152° 51′**; 152° 50.6′.

7. a = **119° 52′**; 119° 52.0′: c = **126° 50′**; 126° 50.7′: β = **115° 29′**; 115° 29.3′.

8. c = **142° 23′**; 142° 23.2′: β = **149° 34′**; 149° 33.7′: γ = **158° 41′**; 158° 40.7′.

9. $a = b = 70° 18.8'$; $\gamma = 102° 13.4'$.

10. $c = 145° 1.3'$; $\beta = 40° 35.7'$; 155° 31.3′.

11. a = **104° 18′**; 104° 17.2′: β = **47° 46′**; 47° 46.1′; γ = **129° 18′**; 129° 18.4′.

12. b = **65° 7′**; 65° 7.7′: α = **139° 12′**; 139° 11.3′; γ = **26° 59′**; 26° 59.5′.

13. b = **58° 31′**; 58° 31.5′: α = **18° 13′**; 18° 13.8′: γ = **114° 21′**; 114° 20.3′.

14. b = **96° 50′**; 96° 49.3′: c = **28° 54′**; 28° 54.9′: α = **132° 21′**; 132° 20.6′.

Exercise 5. Page 231

1. α = **45° 16′**; 45° 14.8′: β = **58° 24′**; 58° 23.6′: γ = **94° 52′**; 94° 52.6′.

2. α = **70° 32′**; 70° 31.4′: β = **47° 20′**; 47° 19.2′: γ = **88° 4′**; 88° 4.4′.

3. a = **124° 50′**; 124° 49.0′: b = **73° 26′**; 73° 25.4′: c = **94° 38′**; 94° 37.4′.

4. a = **70° 56′**; 70° 56.8′: b = **76° 14′**; 76° 14.0′: c = **66° 0′**; 66° 0.2′.

5. α = **72° 46′**; 72° 45.8′: β = **37° 16′**; 37° 16.4′: γ = **105° 16′**; 105° 15.6′:

6. a = **115° 36′**; 115° 34.4′: b = **56° 2′**; 56° 2.0′: c = **96° 52′**; 96° 50.8′.

7. a = **107° 54′**; 107° 53.4′: b = **68° 4′**; 68° 4.2′: c = **64° 10′**; 64° 9.0′.

8. α = **147° 34′**; 147° 35.2′: β = **107° 26′**; 107° 26.2′: γ = **121° 24′**; 121° 23.6′.

9. $\alpha = 48° 35.0'$; $\beta = 68° 36.8'$; $\gamma = 99° 8.2'$.

10. $a = 129° 32.2'$; $b = 103° 7.0'$; $c = 70° 40.0'$.

11. $a = 137° 50.8'$; $b = 80° 58.0'$; $c = 90° 52.2'$.

12. $\alpha = 43° 40.4'$; $\beta = 118° 26.4'$; $\gamma = 66° 8.2'$.

Exercise 6. Page 233

1. $c = \mathbf{78° 46'}$; $78° 45.4'$: $\alpha = \mathbf{105° 36'}$; $105° 36.0'$: $= \mathbf{44° 0'}$; $40° 0.0'$.

2. $c = \mathbf{68° 20'}$; $68° 18.4'$: $\alpha = \mathbf{57° 54'}$; $57° 54.3'$: $= \mathbf{22° 6'}$; $22° 5.1'$.

3. $a = \mathbf{58° 49'}$; $58° 49.2'$: $b = \mathbf{34° 47'}$; $34° 46.4'$: $= \mathbf{36° 22'}$; $36° 22.6'$.

4. $a = \mathbf{95° 26'}$; $95° 25.5'$: $b = \mathbf{43° 24'}$; $43° 23.7'$: $= \mathbf{69° 36'}$; $69° 36.2'$.

5. $a = \mathbf{108° 10'}$; $108° 9.8'$: $\beta = \mathbf{123° 59'}$; $123° 58.7'$: $= \mathbf{72° 43'}$; $72° 42.7'$.

6. $a = \mathbf{123° 31'}$; $123° 30.7'$ $c = \mathbf{84° 47'}$; $84° 47.7'$: $= \mathbf{125° 4'}$; $125° 2.0'$.

7. $b = \mathbf{51° 25'}$; $51° 24.6'$: $c = \mathbf{91° 51'}$; $91° 51.8'$: $= \mathbf{53° 16'}$; $53° 18.8'$.

8. $b = \mathbf{74° 22'}$; $74° 23.8'$: $\alpha = \mathbf{57° 36'}$; $57° 35.8'$: $= \mathbf{115° 8'}$; $115° 7.4'$.

9. $b = 72° 24.0'$; $\alpha = 129° 14.7'$; $\gamma = 68° 35.5'$.

10. $a = 103° 40.8'$; $c = 68° 32.2'$; $\beta = 45° 15.8'$.

11. $b = 85° 20.3'$; $c = 142° 43.3'$; $\alpha = 107° 31.4'$.

12. $a = 108° 54.8'$; $\beta = 103° 0.7'$; $138° 47.3'$.

Exercise 7. Page 235

1. Two. **3.** None. **5.** None. **7.** None.

9. $c = \mathbf{21° 46'}$; $21° 43.6'$: $\alpha = \mathbf{115° 58'}$; $115° 59.2'$: γ $\mathbf{26° 56'}$; $26° 54.4'$.

Or, $c = \mathbf{54° 50'}$; $54° 49.8'$: $\alpha = \mathbf{64° 2'}$; $64° 0.8'$: γ $\mathbf{87° 56'}$; $87° 54.0'$.

10. $a = \mathbf{62° 34'}$; $62° 33.4'$: $\alpha = \mathbf{97° 2'}$; $97° 2.2'$: β $\mathbf{58° 2'}$; $58° 2.1'$.

Or, $a = \mathbf{18° 26'}$; $18° 25.4'$: $\alpha = \mathbf{20° 42'}$; $20° 42.0'$: β $\mathbf{121° 58'}$; $121° 57.9'$.

11. $a = \mathbf{42° 48'}$; $42° 49.0'$: $\alpha = \mathbf{41° 48'}$; $41° 48.4'$: β $\mathbf{38° 20'}$; $38° 20.5'$.

12. $c = \mathbf{46° 14'}$; $46° 15.0'$: $\alpha = \mathbf{45° 38'}$; $45° 36.9'$: γ $\mathbf{39° 58'}$; $39° 57.6'$.

13. No solution.

14. $b = \mathbf{152° 50'}$; $152° 50.4'$: $c = \mathbf{156° 41'}$; $156° 40.8'$: $\beta = \mathbf{136° 22'}$; $136° 21.2'$.

15. No solution. **16.** No solution.

17. $c = \mathbf{24° 28'}$; $24° 28.8'$: $\beta = \mathbf{148° 18'}$; $148° 17.8'$: $\gamma = \mathbf{26° 50'}$; $26° 50.4'$.

18. $a = \mathbf{50° 57'}$; $50° 55.9'$: $b = \mathbf{164° 46'}$; $164° 46.6'$: $\beta = \mathbf{163° 14'}$; $163° 14.4'$.

Or, $a = \mathbf{129° 3'}$; $129° 4.1'$: $b = \mathbf{65° 8'}$; $65° 0.8'$: $\beta = \mathbf{95° 32'}$; $95° 26.0'$.

19. $a = 138° 21.2'$; $\alpha = 137° 58.8'$; $\beta = 78° 41.0'$.

Or, $a = 97° 0.8'$; $\alpha = 91° 5.2'$; $\beta = 101° 19.0'$.

20. $b = 44° 52.5'$; $c = 144° 49.8'$; $\gamma = 139° 41.8'$.

21. $a = 118° 25.8'$; $c = 124° 57.2'$; $\alpha = 104° 1.2'$.

Exercise 8. Page 236

$1.780 \cdot 10^4$ sq. ft. **2.** 8203 sq. ft. **3.** $1.264 \cdot 10^4$ sq. ft. **4.** 8610 sq. ft.

Exercise 9. Page 236

$b = \mathbf{110° 36'}$; $110° 35.8'$: $\alpha = \mathbf{129° 58'}$; $129° 57.2'$: $\gamma = \mathbf{92° 32'}$; $92° 31.2'$.

$\alpha = \mathbf{57° 54'}$; $57° 56.0'$: $\beta = \mathbf{123° 28'}$; $123° 27.0'$: $\gamma = \mathbf{32° 58'}$; $32° 59.4'$.

3. a = **106° 50′**; 106° 48.4′: b = **53° 26′**; 53° 25.2′: c = **73° 26′**; 73° 24.0′.
4. b = **75° 28′**; 75° 26.0′: α = **121° 14′**; 121° 15.0′: β = **64° 16′**; 64° 13.8′.
5. No solution.
6. α = **113° 56′**; 113° 56.4′: β = **85° 2′**; 85° 2.4′: γ = **104° 40′**; 104° 41.4′.
7. a = **155° 2′**; 155° 4.8′: c = **149° 11′**; 149° 11.6′: α = **149° 44′**; 149° 46.0′.
8. a = **55° 55′**; 55° 54.9′: c = **98° 21′**; 98° 20.9′: β = **55° 6′**; 55° 7.0′.
9. b = **58° 51′**; 58° 51.2′: c = **139° 16′**; 139° 16.2′: γ = **132° 10′**; 132° 8.6′.
Or, b = **121° 9′**; 121° 8.8′: c = **60° 26′**; 60° 22.4′: γ = **81° 6′**; 81° 3.6′.
10. b = **113° 2′**; 113° 2.2′: α = **124° 5′**; 124° 3.6′: γ = **71° 29′**; 71° 28.0′.

Exercise 10. Page 240

1. Dep = + 176 mi.; DL = + 163′; lat. = 33° 21′ N.
2. Dep = + 93 mi.; DL = − 128′; lat. = 51° 8′ N.
3. Dep = − 310 mi.; DL = − 9′; lat. = 15° 47′ S.
4. Dep = − 174 mi.; DL = + 198′; lat. = 35° 29′ S.
5. Course = 60° 26′; d = 308 mi.; lat. = 46° 23′ N.
6. Course = 141° 51′; d = 267 mi.; lat. = 25° 6′ N.
7. Course = 325° 34′; d = 424 mi.; lat. = 22° 33′ N.
8. Course = 40° 43′; d = 284 mi.; lat. = 25° 19′ S.
9. Course = 243° 48′; d = 283 mi.; lat. = 19° 44′ S.

Exercise 11. Page 244

1. (23° 15′ N, 80° 48′ W).
2. (53° 16′ N, 6° 6′ E).
3. (40° 58′ N, 59° 19′ W).
4. (50° 44′ N, 74° 11′ E).
5. (19° 55′ S, 161° 10′ E).
6. (15° 54′ S, 151° 8′ W).
7. (64° 11′ N, 139° 37′ W).
8. (31° 45′ N, 88° 53′ E).
9. (22° 59′ S, 151° 48′ E).
10. (17° 40′ S, 131° 19′ E).
11. Course = 319° 22′; d = 179 mi.
12. Course = 227° 0′; d = 153 mi.
13. Course = 94° 59′; d = 598 mi.
14. Course = 257° 11′; d = 870 mi.
15. Course = 252° 12′; d = 1357 mi.
16. Course = 290° 50′; d = 1130 mi.; heading = 283° 20′.

Exercise 12. Page 247

NOTE. The distance, initial course, and bearing of the first city from the second are given in that order.

1. 3144 mi.; 53° 49′; 291° 53′.
2. 6337 mi.; 300° 18′; 47° 40′.
3. 4755 mi.; 306° 0′; 55° 35′.
4. 3346 mi.; 299° 26′; 87° 0′.
5. 4151 mi.; 300° 34′; 45° 32′.
6. 1617 mi.; 35° 13′; 223° 23′.
7. 2223 mi.; 260° 22′; 61° 15′.
8. 4139 mi.; 217° 14′; 32° 3′.
9. 6447 mi.; 55° 46′; 240° 18′.
10. 5226 mi.; 55° 29′; 327° 57′.
11. 2229 mi.; 69° 44′; 281° 34′.
12. 2125 mi.; 273° 34′; 65° 44′.

13. 1343 mi.; 266° 55′; 58° 37′.
14. 2714 mi.; 214° 17′; 36° 16′.
15. 4039 mi.; 59° 26′; 323° 23′.
16. 597 mi.; 88° 22′; 281° 33′.
17. 2466 mi.; 46° 30′; 266° 16′.
18. 1619 mi.; 14° 3′; 201° 17′.
19. 2774 mi.; 308° 30′; 69° 35′.
20. 4267 mi.; 26° 43′; 325° 36′.
21. 4047 mi.; 34° 31′; 310° 21′.
22. 1125 mi.; 106° 39′; 294° 47′.
23. Great circle distance is 16 miles shorter.
24. Great circle distance is 12 miles shorter.

Exercise 13. Page 252

1. 8:36 A.M.
2. 2:41 P.M.
3. 5:07 A.M.; bearing = 70° 41′.
4. 7:02 P.M.; bearing = 293° 51′.
5. Longest, 21 hr. and 0 min.; bearing = 20° 33′.
Shortest, 3 hr. and 0 min.; bearing = 159° 27′.
6. Longest, 13 hr. and 56 min.; bearing = 62° 40′.
Shortest, 10 hr. and 4 min.; bearing = 117° 20′.
7. 47° 48′.
8. 14° 19′.
9. 53° 52′.
10. – 21° 40′.
11. Alt. = 54° 48′; az. = 120° 58′.
12. Alt. = 50° 27′; az. = 54° 35′.
13. Alt. = 26° 25′; az. = 345° 44′.
14. Alt. = 18° 32′; az. = 249° 21′.
15. Alt. = 60° 26′; az. = 79° 25′.

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